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Generalized Bernstein Inequalities

By SAMUEL KARLIN in Stanford (California, USA)

Dedicated to my friend, colleague and teacher Professor Gábor Szegő on his 70th birthday

§ 1. Introduction

Two celebrated theorems due to BERNSTEIN and MARKOFF describe extremal characterizations of the Tchebycheff polynomials. The origin of these developments stems from interest in the following problems.

Problem 1. Determine the polynomial of degree $m-1$ which maximizes

$$\max_{-1 \leqq x \leqq 1} |P_{m-1}(x)| \tag{1}$$

among all polynomials of degree $m-1$ satisfying the conditions

$$(1-x^2)P_{m-1}^2(x) \leqq 1 \qquad (-1 \leqq x \leqq 1). \tag{2}$$

Problem 2. Let $P_m(x)$ denote any polynomial of degree m obeying the restriction

$$\max_{-1 \leqq x \leqq 1} |P_m(x)| \leqq 1.$$

Find an upper bound for

$$\max_{-1 \leqq x \leqq 1} |P'_m(x)|.$$

The extremal polynomial in each case turns out to be a classical Tchebycheff polynomial. The solutions to Problems 1 and 2 lead to what is known as the Markoff—Bernstein inequalities.

The usual method of analyzing Problem 2 is to reduce it to Problem 1. It is customary to first formulate a trigonometric version of Problem 2 which is easily solved. It is then possible to combine the result of the trigonometric case with the conclusion of Problem 1 and thereby uncover the solution of Problem 2 (for the details of this method see PÓLYA—SZEGŐ [14, page 90]).

The extremal characterization and the uniqueness proof for Problem 1 depend on the existence of a polynomial which exhibits maximum oscillation under the constraint (2). In the classical case this polynomial is $U_{m-1}(t)$, the Tchebycheff polynomial of the second kind. The essential fact duly exploited in the proof is that any other polynomial $P_{m-1}(t)$ meeting the constraint (2) cannot provide a larger value in (1) or otherwise the difference $P_{m-1}^2(t) - U_{m-1}^2(t)$ has too many zeros.

Extensions, refinements and elaborations of Problems 1 and 2 have moved in several directions. In most instances only Problem 2 has been generalized. For example:

1. The result of Problem 2 has been suitably extended to certain classes of entire functions of order ϱ, e. g., BOAS [6, Chap. 8], ACHIESER [1, p. 140], BERNSTEIN [2].

2. An extensive accounting of aspects of Problem 2 when the domain of definition of $P_n(z)$ is enlarged so that z ranges over a region of the complex plane is available (see SZEGŐ [20] and BERNSTEIN [5] and DOCEV [7]).

3. A multivariate generalization operates in terms of harmonic polynomials where the derivative function of Problem 2 is replaced by a gradient expression (see e. g., SZEGŐ [15]). In HÖRMANDER [8] these considerations are related to certain concepts of hyperbolic cones.

We propose a different kind of generalization with Problem 1 as the point of departure. The novelty of these generalizations is to replace polynomials $P_n(t)=\sum_{i=0}^{n} a_i t^i$ formed from the system of functions $\{t^i\}_{i=0}^n$ by u-polynomials generated as linear combinations of a Tchebycheff system of functions $\{u_i\}_0^n$.

Let $u_0(t), \ldots, u_n(t)$ be continuous functions on a finite interval $[a, b]$. These functions are called a Tchebycheffian system or T-system provided all the determinants

$$U\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & u_1(t_0) & \ldots & u_n(t_0) \\ u_0(t_1) & u_1(t_1) & \ldots & u_n(t_1) \\ \vdots & \vdots & & \vdots \\ u_0(t_n) & u_1(t_n) & \ldots & u_n(t_n) \end{vmatrix} \tag{3}$$

for arbitrary choices of $\{t_i\}_{i=0}^n$ satisfying

$$a \leqq t_0 < t_1 < \ldots < t_n \leqq b$$

maintain a single strict sign. Without restricting generality (multiply $u_n(t)$ suitably by $+1$ or -1), we may assume the determinants in (3) are positive.

In the particular case $u_i(t)=t^i$ $(i=0, \ldots, n)$, (3) reduces to the familiar Vandermonde determinant.

Tchebycheff systems occur naturally in various domains of mathematics. For example, GANTMACHER and KREIN [9] establish that for regular Sturm—Liouville eigenvalue problems with discrete positive spectrum the first $n+1$ eigenfunctions $\varphi_0, \varphi_1, \ldots, \varphi_n$ constitute a T-system. More generally the first $n+1$ eigenfunctions associated with an integral transformation

$$T\varphi = \int_a^b K(x, y)\varphi(y)\, d\sigma(y) \qquad (d\sigma \geqq 0),$$

where $[a, b]$ is finite and K has an iterate which is strictly totally positive, i. e., satisfies certain determinantal inequalities, form a T-system.

T-systems play a role in interpolation problems, moment theory, the study of oscillation properties of polynomials and in other branches of analysis. For a geometrical study of T-systems, the reader may consult KREIN [12] or a forthcoming book [11] by the author and W. STUDDEN.

If the functions $u_i(t)$ $(i=0, 1, \ldots, n)$ are sufficiently differentiable we sometimes extend the definition of

$$U\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix}$$

given in (3) to allow for equalities amongst the t_i. Thus, if $a \leqq t_0 \leqq t_1 \leqq \ldots \leqq t_n \leqq b$ then

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix}$$

is defined as the determinant of (3) where for each set of equal t_i we replace successive rows by their successive derivatives. For example if $a \leqq t_0 = t_1 = \ldots = t_{k-1} < t_k < \ldots \ldots < t_{n-2} < t_{n-1} = t_n \leqq b$, then

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} = \begin{vmatrix} u_0(t_0) & \cdots & u_n(t_0) \\ u_0^{(1)}(t_0) & \cdots & u_n^{(1)}(t_0) \\ \vdots & & \vdots \\ u_0^{(k-1)}(t_0) & \cdots & u_n^{(k-1)}(t_0) \\ u_0(t_k) & \cdots & u_n(t_k) \\ \vdots & & \vdots \\ u_0(t_{n-2}) & \cdots & u_n(t_{n-2}) \\ u_0(t_n) & \cdots & u_n(t) \\ u_0'(t_n) & \cdots & u_n'(t_n) \end{vmatrix}.$$

The system $\{u_i(t)\}_0^n$ will be called *extended Techebycheffian* of order r (abbreviated ET_r) provided $u_i(t)$ are of class C^{r-1} and

$$U^*\begin{bmatrix} t_0, \ldots, t_n \\ 0, \ldots, n \end{bmatrix} > 0$$

for all $t_0, \ldots, t_n$ satisfying $a \leqq t_0 \leqq t_1 \leqq \ldots \leqq t_n \leqq b$, where equalities are permitted in groups consisting of at most r successive t values.

In the following, the term *polynomial* will refer to a function of the form $u(t) = \sum_{i=0}^{n} a_i u_i(t)$, where the a_i $(i=0, \ldots, n)$ are real constants and the functions $u_i(t)$ $(i=0, 1, \ldots, n)$ constitute either a T-system or an ET-system on a closed interval $[a, b]$. By the *index* of a set $\{t_1, \ldots, t_k\}$ for $t_i \in [a, b]$ $(i=1, \ldots, k)$, we shall mean the number of distinct points in this set under the special convention that the endpoints a and b are counted as one-half while interior points are given a count of one. For example the set $\{a, (a+b)/2\}$ has index 3/2 and the set $\{a, (a+b)/2, b\}$ has index 2.

Before introducing our main theorems we present one lemma due essentially to Krein [12] which provides information concerning the structure of polynomials and the nature of their zeros. For any polynomial u, t_0 is said to be a non-nodal zero if $u(t_0)=0$ and $u(t) \leqq 0$ or $u(t) \geqq 0$ for t in some open neighborhood of t_0. All other zeros including a and b are called nodal. The symbol $Z(u)$ denotes the number of zeros of u where nodal zeros are counted once and non-nodal zeros twice. The maximum number of sign changes in the sequence $\{a_i\}_1^m$, where zero terms can be counted as either plus or minus, is denoted by $V(a_1, a_2, \ldots, a_m)$.

We have

Lemma 1. (a) *If* $\{u_i(t)\}_0^n$ *is a T-system on* $[a, b]$ *and* $u(t)=\sum_{i=0}^{n} a_i u_i(t) \not\equiv 0$ *is a polynomial, then*

(i) $V(u(t_0), \ldots, u(t_{n+1})) \leqq n$ *for all* t_i *satisfying* $a \leqq t_0 < \ldots < t_{n+1} \leqq b$,

(ii) $Z(u) \leqq n$.

(b) *If* $\{u_i(t)\}_0^n$ *is an* ET_r*-system then any polynomial possesses at most n zeros (counting multiplicities up to order r). The number of zeros of a polynomial u by this counting procedure is denoted by* $Z_r^*(u)$.

The proof of Lemma 1 is simple; a formal argument appears in [11].

The key tool of this paper is the representation theorem expressing a positive polynomial as a unique combination of two other polynomials exhibiting special oscillation properties. Various extremal problems are solved by suitably invoking the representation theorem. In this category we include a number of cases of best minimax approximation, the Markoff—Bernstein inequalities for polynomials and extremal problems of a type introduced by BERNSTEIN in [4].

We state the key fact established in [10] which underlies most of the developments of this paper.

Theorem A. *If* $\{u_i(t)\}_0^n$ *is a T-system and* $p(t)$ *and* $q(t)$ *are continuous functions on* $[a, b]$ *such that there exists a polynomial* $v(t)$ *with* $p(t) > v(t) \geqq q(t)$ *then there exists exactly two polynomials* $\underline{u}(t)$ *and* $\bar{u}(t)$ *satisfying the following properties:*

(i) $p(t) \geqq u(t) \geqq q(t)$ *and* $v-u$ *vanishes at n interior points,*

(ii) *there exists* $n+1$ *points* $s_1 < s_2 < \ldots < s_{n+1}$ *which interlace the zeros of* $v-u$ *such that for* $n=2m$

$$\bar{u}(s_i) = \begin{Bmatrix} p(s_i), & i \text{ odd} \\ q(s_i), & i \text{ even} \end{Bmatrix}, \quad \underline{u}(s_i) = \begin{Bmatrix} p(s_i), & i \text{ even} \\ q(s_i), & i \text{ odd} \end{Bmatrix}$$

and for $n = 2m+1$

$$\underline{u}(s_i) = \begin{Bmatrix} p(s_i), & i \text{ odd} \\ q(s_i), & i \text{ even} \end{Bmatrix}, \quad \bar{u}(s_i) = \begin{Bmatrix} p(s_i), & i \text{ even} \\ q(s_i), & i \text{ odd} \end{Bmatrix}.$$

When p and q are polynomials then $\bar{u}$ *can be distinguished from* $\underline{u}$ *in that* $\bar{u}(b)=0$ *while* $\underline{u}(b) \neq 0$.

Corollary. *If p is a polynomial then* $p-\underline{u}$ *and* $p-\bar{u}$ *vanish on a set of index* $n/2$.

There are many extremal problems in the theory of approximation of functions by polynomials whose solutions are intimately connected with the special polynomials $\underline{u}$ and $\bar{u}$ for appropriate choices of p and q. In several natural examples discussed later we will find that the extremal polynomial often coincides with $\underline{u}$ or $\bar{u}$. The validity for this result rests on a simple counting principle which exploits the special oscillation properties of $\bar{u}$ and $\underline{u}$. Actually, since the polynomials $\underline{u}$ and $\bar{u}$ cover the distance between $p(t)$ and $q(t)$ at least n times as t traverses $[a, b]$ it is clear that if $u(t)$ is an arbitrary polynomial lying between p and q then $\bar{u}-u$ and $\underline{u}-u$ exhibit at least n

zeros under the convention that non-nodal zeros are counted twice. But Lemma 1 tells us that $\bar{u}-u$ and similarly $\underline{u}-u$ cannot possess $n+1$ zeros without vanishing identically. This requires that $\bar{u}-u$ or $\underline{u}-u$ obey certain inequalities. Such inequalities can be interpreted as extremal characterizations of $\bar{u}$ and $\underline{u}$ corresponding to certain variational problems.

§ 2. Extremal Problems

Unless stated otherwise we assume throughout this section that $\{u_i(t)\}_{i=0}^n$ constitutes an ET_2-system.

Let $p(t)$ be a continuous function on $[a, b]$ and $q(t)$ a fixed *polynomial* satisfying $p(t) > q(t)$ for all $t \in [a, b]$. Consider the class of polynomials,

$$\mathcal{U} = \{u | q(t) \leqq u(t) \leqq p(t),\ t \in [a, b];\ u(a) = q(a)\}. \tag{4}$$

Theorem A (see [10]) and its corollary affirm the existence of a unique polynomial $v_* \in \mathcal{U}$ characterized by

Property A: $v_* - q$ vanishes on a set of index $n/2$ and $p - v_*$ vanishes at least once between each pair of zeros of $v_* - q$ and between the largest interior zero and the endpoint b. The special polynomial v_* is, in fact, the polynomial $\bar{u}$ when n is even and $\underline{u}$ when n is odd. Moreover, v_* enjoys several remarkable extremal properties as attested to by Theorems 1 and 2 which follow.

Theorem 1. *Let $\mathcal{U}$ be defined as in* (4) *and let v_* be the unique polynomial characterized by Property* A. *Then*

$$\max_{u \in \mathcal{U}} u'(a) \tag{5}$$

is attained uniquely in $\mathcal{U}$ by the polynomial v_*.

Proof. Since the polynomials in $\mathcal{U}$ are uniformly bounded, we easily infer that $\mathcal{U}$ is a compact family of polynomials. Hence the maximum in (5) is attained. If a polynomial w attains this maximum then $w'(a) \geqq v_*'(a)$.

Let s_1 be the first zero of $p(t) - v_*(t)$. Clearly $w(t) - v_*(t)$ possesses at least $n-1$ zeros on $[s_1, b]$ with the convention that non-nodal zeros are counted twice. If $w'(a) > v_*'(a)$ then $w(t) - v_*(t)$ has a zero in $(a, s_1]$ which together with the endpoint a and the $n-1$ zeros in $[s_1, b]$ provide a total of $n+1$ zeros. If $w'(a) = v_*'(a)$ then $w(t) - v_*(t)$ exhibits a zero at $t = a$ of multiplicity at least two. In either case $w(t) - v_*'(t)$ has $n+1$ zeros where multiple zeros are counted twice. It follows from Lemma 1 that $w(t) \equiv v_*(t)$.

We next introduce a class of polynomials slightly more restricted than (4), namely

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & n \text{ odd,} \\ \{u | u \in \mathcal{U},\ u(b) = q(b)\} & n \text{ even.} \end{cases} \tag{6}$$

Assume p is of class C^1 and that

(i) for n even the function

$$f(t) = \frac{p(t)-q(t)}{(b-t)(t-a)}$$

is strictly decreasing on (a, t_0) and strictly increasing on (t_0, b) for some t_0 in (a, b),

(ii) for n odd the function

$$g(t) = \frac{p(t)-q(t)}{t-a}$$

is strictly decreasing on $[a, b]$.

The functions $f(t)$ and $g(t)$ as well as the expressions in (7) and (8) below are extended by continuity where the denominators vanish.

Theorem 2 (Generalized Bernstein—Markoff Inequality). *Let the assumptions stated in* (i) *and* (ii) *prevail. The polynomial* $v_* \in \mathscr{U}_0$, *explicitly characterized by Property* A, *uniquely attains*

$$\max_{u\in\mathscr{U}_0} \max_{t\in[a,b]} \frac{u(t)-q(t)}{(b-t)(t-a)} \tag{7}$$

if n *is even and*

$$\max_{u\in\mathscr{U}_0} \max_{t\in[a,b]} \frac{u(t)-q(t)}{(t-a)} \tag{8}$$

if n *is odd.*

Remark. The relation of this theorem and Problem 1 is made explicit in Section 5 where other applications are also indicated.

Proof. We deal only with (7). Note that the special polynomial v_* obeys the property that

$$\frac{v_*(t)-q(t)}{(b-t)(t-a)} \tag{9}$$

oscillates between 0 and $f(t)$ $(a<t<b)$ in a manner that it equals $f(t)$ at m $(n=2m)$ points $s_1, \ldots, s_m$ and equals zero at $m-1$ points $t_1, \ldots, t_{m-1}$ which together satisfy $a<s_1<t_1<s_2<t_2<\ldots<t_{m-1}<s_m<b$. From the monotonicity properties of $f(t)$ ((i) and (ii) above) it follows that the maximum in (7) is achieved for some t in $[a, s_1]\cup[s_m, b]$. Now if $w\in\mathscr{U}_0$ attains (7) and $w\neq v_*$ we infer on the basis of Theorem 1 that

$$\frac{w'(a)-q'(a)}{b-a} < \frac{v_*'(a)-q'(a)}{b-a}.$$

By reversing the interval $[a, b]$ Theorem 1 also implies that

$$\frac{w'(b)-q'(b)}{b-a} > \frac{v_*'(b)-q'(b)}{b-a},$$

so that
$$\frac{w(x_0)-q(x_0)}{(b-x_0)(x_0-a)} \geqq \frac{v_*(x_0)-q(x_0)}{(b-x_0)(x_0-a)}$$

for some $x_0\in(a, s_1]\cup[s_m, b)$. However, in this case $w-v_*$ has $n-1$ zeros on (a, b)

counting non-nodal zeros twice so that $w-v_*$ possesses $n+1$ zeros on $[a, b]$ and hence $w \equiv v_*$.

The demonstration that v_* is the only polynomial attaining (8) is accomplished by analogous reasoning. The proof of the theorem is complete.

In Theorem 1 it was shown that the polynomial v_* characterized by Property A possesses the maximum derivative at the end point a. This same property of v_* actually holds for an arbitrary $z \in [a, b]$ in a sense that we now describe.

For n odd, say $n = 2m+1$, there exist points $a = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \tilde{s}_2 < \ldots < \tilde{t}_m < \tilde{s}_{m+1} \leqq b$ at which the polynomial v_* satisfies the relations

$$v_*(\tilde{t}_k) = q(\tilde{t}_k) \qquad (k = 0, 1, \ldots, m)$$

and

$$v_*(\tilde{s}_k) = p(\tilde{s}_k) \qquad (k = 1, 2, \ldots, m+1).$$

For $n = 2m+2$ the only modification is that $\tilde{s}_{m+1} < b$ and $v_*(b) = q(b)$. As t traverses an interval of the type $(\tilde{t}_i, \tilde{s}_{i+1})$ the polynomial v_* extends from the value $q(\tilde{t}_i)$ to the value $p(\tilde{s}_{i+1})$ while on an interval of the type $(\tilde{s}_i, \tilde{t}_i)$ the values of $v_*(t)$ vary from $p(\tilde{s}_i)$ to $q(\tilde{t}_i)$. Generally v_* is increasing on $(\tilde{t}_i, \tilde{s}_{i+1})$ and decreasing on $(\tilde{s}_i, \tilde{t}_i)$.

Bearing this in mind we define the sets A and B as follows.

$$A = (\tilde{t}_0, \tilde{s}_1) \cup (\tilde{t}_1, \tilde{s}_2) \cup \ldots \cup (\tilde{t}_m, \tilde{s}_{m+1}),$$

$$B = \begin{cases} (\tilde{s}_1, \tilde{t}_1 \cup (\tilde{s}_2, \tilde{t}_2) \cup \ldots \cup (\tilde{s}_m, \tilde{t}_m) & (n = 2m+1), \\ (\tilde{s}_1, \tilde{t}_1) \cup (\tilde{s}_2, \tilde{t}_2) \cup \ldots \cup (\tilde{s}_m, \tilde{t}_m) \cup (\tilde{s}_{m+1}, b) & (n = 2m+2). \end{cases}$$

For each fixed $z \in [a, b]$ let $\mathcal{U}(z)$ be the class of polynomials

$$\mathcal{U}(z) = \{u | q(t) \leqq u(t) \leqq p(t), t \in [a, b], u(z) = v_*(z)\}.$$

Theorem 3. *The polynomial v_* uniquely attains*

$$\max_{u \in \mathcal{U}(z)} u'(z) \quad \text{if} \quad z \in A, \tag{10}$$

$$\min_{u \in \mathcal{U}(z)} u'(z) \quad \text{if} \quad z \in B. \tag{11}$$

The proof is similar to that of Theorem 2 and is therefore omitted.

Remark. We emphasize again that if $\{u_i\}_0^n$ is not an ET-system of order 2 but simply a T-system, the preceding theorems remain in force except for the uniqueness assertion.

§ 3. Generalized Bernstein—Markoff Inequalities for Infinite Intervals

The result of Theorem 2 may easily be extended to the semi-infinite interval $[0, \infty)$. We assume that $\{u_i\}_0^n$ is an ET-system of order 2 on $[0, \infty)$ and in addition we impose the following requirements:

(i) $u_n(t) > 0$, $t \geqq \hat{t}$ for some $\hat{t} > 0$,

(ii) $\lim_{t \to \infty} u_i(t)/u_n(t) = 0$, $i = 0, \ldots, n-1$,

(iii) $\{u_i\}_0^{n-1}$ is a T-system on $[0, \infty)$.

Consider a continuously differentiable function $f(t)>0$ on $[0,\infty)$ satisfying the condition that

$$\lim_{t\to\infty}\frac{f(t)}{u_n(t)} \tag{12}$$

exists where its value is positive or plus infinity. Notice that these assumptions include the case where $u_i(t)=t^i$ $(i=0,1,\dots,n)$ and $f(t)=e^t$.

We wish to determine the polynomial $\tilde{u}$ which attains

$$\max_{u\in A}\max_{t\in[0,\infty]}\frac{u(t)}{tf(t)} \tag{13}$$

where A is the class of polynomials defined by the conditions

$$A=\{u\,|\,0\leqq u(t)\leqq f(t),\,u(0)=0\}$$

and the value $t^{-1}u(t)$ is defined to be $u'(0)$ for $t=0$.

To solve this problem we first determine that polynomial (as in Theorem 1) which attains

$$\max_{u\in A}u'(0). \tag{14}$$

As in the case of the finite interval the extremal polynomials which yield the maximums in (13) and (14) agree.

Let $w(t)$ be a strictly positive function on $[0,\infty)$ such that $w(t)=u_n(t)$, $t\geqq\hat{t}$ and set

$$v_k(x)=\begin{cases}\dfrac{u_k(\tan x)}{w(\tan x)} & (x\in[0,\pi/2)),\\ \delta_{kn} & (x=\pi/2),\end{cases}$$

$$\bar{f}(x)=\frac{f(\tan x)}{w(\tan x)}\qquad(x\in[0,\pi/2]).$$

The system $\{v_k\}_0^n$ is a T-system on $[0,\pi/2]$ and $\bar{f}(x)>0$ on $[0,\pi/2)$ and its value at $\pi/2$ is positive or possibly infinite because of (12).

Suppose first that $\bar{f}(\pi/2)$ is finite. Let v_* be the polynomial satisfying Property A. If $\bar{f}(\pi/2)$ is infinite we construct v_* first for

$$\bar{f}_N(x)=\begin{cases}\bar{f}(x) & (\bar{f}(x)\leqq N),\\ N & (\bar{f}(x)>N)\end{cases}$$

observing that the same polynomial v_* occurs for all N sufficiently large.

We now transform the polynomial v_* according to

$$u_*(t)=w(t)v_*(\tan^{-1}t)\qquad(t\in[0,\infty)). \tag{15}$$

For $n=2m$ the polynomial $u_*(t)$ vanishes at the points

$$0=\tilde{t}_0<\tilde{t}_1<\ldots<\tilde{t}_{m-1}$$

and agrees with $f(t)$ at the points $\tilde{s}_1,\tilde{s}_2,\dots,\tilde{s}_m$. Also, these values interlace in the manner

$$0 = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \ldots < \tilde{t}_{m-1} < \tilde{s}_m < \infty$$

The corresponding points for the case $n = 2m+1$ have the form

$$0 = \tilde{t}_0 < \tilde{s}_1 < \tilde{t}_1 < \tilde{s}_2 < \ldots < \tilde{s}_m < \tilde{t}_m < \tilde{s}_{m+1} < \infty.$$

Note that in the even case we have $2m-1$ zeros counting non-nodal zeros twice. An additional zero occurs at plus infinity and necessarily $v_*(t) = \alpha_0 v_0(t) + \ldots + \alpha_{n-1} v_{n-1}(t)$, the coefficient of $v_n(t)$ being zero since v_* vanishes at $\pi/2$.

Theorem 4. *The polynomial u_* defined by* (15) *uniquely attains*

$$\max_{u \in A} u'(0)$$

and

(16)
$$\max_{u \in A} \max_{t \in [0, \infty]} \frac{u(t)}{tf(t)}$$

where
$$A = \{u | 0 \leqq u(t) \leqq f(t),\ u(0) = 0\}.$$

The proof of this theorem parallels the preceding analysis. Note that $u(t)/f(t) \leqq 1$ and that the function $1/t$ is decreasing and so the maximum in (16) is achieved in the interval $(0, \tilde{s}_1)$. From here on the argument proceeds as in Theorem 2.

§ 4. Generalized Bernstein—Markoff Inequalities for Periodic Functions

In this section we develop some periodic versions of the generalized Bernstein—Markoff type inequalities.

As is natural to the circle case we restrict n to be even, say $n = 2m$, and assume that $\{u_i\}_0^{2m}$ constitutes an ET system of order 2 consisting of periodic functions (see [11]). An interval of periodicity is assumed to be of length $b-a$. We sometimes require that $\{u_i\}_0^{2m}$ be ET of order 3. The extended Tchebycheffian assumptions of order 2 and 3 are imposed in order to assure unique solutions to various extremal problems.

The following basic theorem proved in [10] is the counterpart of Theorem A.

Theorem B. *Let $p(t)$ be a positive, periodic and continuous function of period length $b-a$. For a fixed $t_0 \in [a, b)$, let $v(t; t_0)$ represent the unique polynomial constructed in Theorem 6 of* [10] *possessing the properties:*

(i) $p(t) \geqq v(t; t_0) \geqq 0$,

(ii) $v(t; t_0)$ *has m distinct zeros one of which is t_0,*

(iii) $p(t) - v(t; t_0)$ *vanishes at least once between each pair of zeros of $v(t; t_0)$ (viewed in the periodic sense).*

Consider the class of polynomials

$$\mathcal{W}(t_0) = \{u | 0 \leqq u(t) \leqq p(t),\ t \in [a, b),\ u(t_0) = 0\}.$$

An extremal characterization of $v(t; t_0)$ is embodied in the following result. (Compare with Theorem 1.)

Theorem 5. *If* $\{u_i\}_0^{2m}$ *is an ET-system of order* 3 *then*

$$\max_{u\in\mathscr{W}(t_0)} u''(t_0)$$

is uniquely attained by the polynomial $v(t; t_0)$.

Proof. Let w be any polynomial in $\mathscr{W}(t_0)$ for which

$$w''(t_0) = \max_{u\in\mathscr{W}(t_0)} u''(t_0).$$

Then $w''(t_0)\geqq v''(t_0; t_0)$. If $w''(t_0)>v''(t_0; t_0)$ the function $w(t)-v(t; t_0)$ has at least $2m+1$ zeros counting multiple zeros twice. If $w''(t_0)=v''(t_0; t_0)$ then $w(t)-$ $-v(t; t_0)$ has at least $2m+1$ zeros where the zero at t_0 is of order 3. In both contingencies we contradict the assumptions on the system $\{u_i\}_0^{2m}$ unless $w(t)\equiv v(t; t_0)$.

Our next objective concerns the formulation of the analog of Theorem 3. Let $p(t)$ and $q(t)$ be two continuous periodic functions on $[a, b)$ for which $p(t)>q(t)$ and suppose there exists a polynomial $\tilde{u}$ satisfying $p(t)>\tilde{u}(t)>q(t)$. For each $t_0\in[a, b)$, Theorem 7 of [10] affirms the existence of a unique polynomial $u(t; t_0)$ possessing the properties

(i) $p(t)\geqq u(t; t_0)\geqq q(t)$,
(ii) $\tilde{u}(t_0)=u(t_0; t_0)$ and there exists n points $\{s_i\}_1^n$, $s_1<t_0<s_2<\ldots<s_n<s_1+b-a$ such that and $u(t; t_0)$ equals $q(t)$ and $p(t)$ alternately at $s_1, s_2, \ldots, s_n$.

An example of the class of polynomials $u(t; t_0)$ is given later in the case of trigonometric polynomials where $p(t) = -q(t)=a$ positive polynomial $h(t)$ of order at most m. The oscillation properties of $u(t; t_0)$ are basic to the solution of certain extremal problems as described in Theorem 6 below. In order to prepare for Theorem 6, we note some preliminaries.

Suppose we specify a point $z_0\in[a, b)$ and a value c satisfying $q(z_0)<c<p(z_0)$. Since the coefficients of $u(t; t_0)$ are continuous functions of the parameter t_0 we deduce the existence of two polynomials $\bar{u}$ and $\underline{u}$ each equal to c at the point z_0 satisfying conditions (i) and (ii) except that $\bar{u}$ and $\underline{u}$ alternate in the opposite direction. Specifically,

$$\underline{u}(s_1)=p(s_1) \quad \text{and} \quad \underline{u}(s_2)=q(s_2), \quad \text{etc.}$$

while

$$\bar{u}(s_1)=q(s_1) \quad \text{and} \quad \bar{u}(s_2)=p(s_2), \quad \text{etc.}$$

The sets of points $\{s_i\}$ associated with the two polynomials $\bar{u}$ and $\underline{u}$ will in general differ.

Consider the class of polynomials $\mathscr{V}$ defined by

$$\mathscr{V}(z_0)=\{u\,|\,q(t)\leqq u(t)\leqq p(t),\ t\in[a, b),\ u(z_0)=c,\ q(z_0)<c<p(z_0)\}.$$

Lemma 2. *If* $\{u_i\}_0^{2m}$ *is a periodic ET-system of order* 2 *then*

$$\max_{u\in\mathscr{V}(z_0)} u'(z_0) = \bar{u}'(z_0)$$

and

$$\min_{u\in\mathscr{V}(z_0)} u'(z_0) = \underline{u}'(z_0).$$

In each case the extremal polynomial is unique.

Proof. The proof again is accomplished by appropriately counting zeros and using the fact that $\bar{u}$ and $\underline{u}$ oscillate a maximum number of times between $p(t)$ and $q(t)$. We omit the details.

We are now in possession of the ingredients necessary to prove the principal theorem of this section. The admissible class of polynomials consists of

$$\mathscr{V}_h = \{u \mid |u(t)| \leqq h(t)\}$$

where $h(t)$ is a strictly positive continuous periodic function on $[a, b)$.

Theorem 7. *Let $\{u_i\}_0^{2m}$ be a periodic ET-system of order 2. The value*

$$\max_{u \in \mathscr{V}_h} \max_{t \in [a,b)} |u'(t)|$$

is attained by a polynomial $u(t;\, t_0)$ for some t_0. In other words, in computing the maximum it is enough to restrict attention to the one parameter family of polynomials $u(t;\, t_0)$.

Proof. Consider any member $u \in \mathscr{V}_h$ and fix a point $t^* \in [a, b)$. We distinguish two cases according as $|u(t^*)| < h(t^*)$ or $|u(t^*)| = h(t^*)$. If the first possibility prevails then we appeal to Lemma 2 which affirms the existence of a polynomial $u(t;\, t_0) \in \mathscr{V}_h$ for some t_0 with the properties

$$u(t^*;\, t_0) = u(t^*) \quad \text{and} \quad |u'(t^*, t_0)| \geqq |u'(t^*)|. \tag{17}$$

In the second case when $|u(t^*)| = h(t^*)$ we increase the function h slightly to h_ε creating the situation $|u(t^*)| < h_\varepsilon(t^*)$. The polynomials $u_\varepsilon(t;\, t_0)$ and their derivatives $u'_\varepsilon(t, t_0)$ vary continuously with ε uniformly in t. (This is so since the coefficients are continuous functions of ε and t_0.) We can now appeal to the preceding case and deduce again the validity of (17) with $u(t^*;\, t_0)$ replaced by $u_\varepsilon(t^*;\, t_0)$ (here t_0 may also depend on ε). Invoking the standard limiting process on ε we infer in all circumstances the validity of (17).

The second relation of (17) can be expressed in the form

$$\sup_{t_0} |u'(t^*;\, t_0)| \geqq \sup_{u \in \mathscr{V}_h} |u'(t^*)|.$$

Since t^* is arbitrary in $[a, b)$, the assertion of the theorem is established.

§ 5. Examples

We begin with two examples of Theorem 7.

Example 1. Consider the T-system of trigonometric functions

$$1,\ \cos\theta,\ \sin\theta,\ \cos 2\theta,\ \sin 2\theta,\ \ldots,\ \cos m\theta,\ \sin m\theta. \tag{18}$$

The special polynomials $u(\theta;\, \theta_0)$ oscillating between $p(\theta) \equiv 1$ and $q(\theta) \equiv -1$ asserted in Theorem B are $\sin m(\theta + \theta_0)$. Application of Theorem 7 and Lemma 2 gives the following classical result. If $g(\theta)$ is a trigonometric polynomial of degree m and

$$|g(\theta)| \leqq 1$$

then

$$|g'(\theta)| \leqq m,$$

with equality if and only if $g(\theta) = \sin m(\theta + \theta_0)$.

Example 2. The T-system under consideration is again (18). Let $h(\theta)$ be a fixed positive trigonometric polynomial of degree $l \leqq m$ and let

$$h(\theta) = |g(z)|^2$$

where $z = e^{i\theta}$ and

$$g(z) = \gamma \prod_{\nu=1}^{l} (z - z_\nu) \qquad (\gamma > 0, |z_\nu| < 1, \nu = 1, \dots, l).$$

The special polynomials of Theorem B lying between $h(\theta)$ and $-h(\theta)$ of maximum oscillation can be explicitly determined. In fact the family of polynomials

$$u(\theta; \theta_0) = \left(\Re e^{i\psi(\theta_0)} z^{m-2l} [g(z)]^2\right), \; z = e^{i\theta}, \; 0 \leqq \theta < 2\pi,$$

where ψ is a real parameter depending on θ_0 ($\Re$ = real part), fulfills the requirements

$$-h(\theta) \leqq u(\theta; \theta_0) \leqq h(\theta)$$

and $u(\theta; \theta_0)$ touches $h(\theta)$ and $-h(\theta)$ alternately m times. The formal proof of this fact appears in [11], see also SZEGŐ [18]. These polynomials coincide with the class $u(\theta; \theta_0)$ described in Theorem B.

We apply Theorem 7 as follows.

Suppose $P(\theta)$ is a trigonometric polynomial of order at most m satisfying

$$|P(\theta)| \leqq \gamma^{-2} h(\theta).$$

Then the value

$$\max_P \max_\theta |P'(\theta)| \tag{19}$$

is achieved by a polynomial of the form

$$u(\theta; \varphi) = \gamma^{-2} \Re[e^{i\varphi} z^{m-2l} (g(z))^2] \quad (z = e^{i\theta}),$$

where φ is a parameter.

In the special case where $h(\theta) = 1 - 2r \cos\theta + r^2$ ($|r| < 1$) a few elementary calculations show that the value of (19) is $(1 + |r|)^2 (m + |r|(m-2))^2$.

We next turn to

Example 3. All the results of this example are classical, however, it is instructive to fit them into the framework of the previous sections.

We start with the representation

$$1 = T_m^2(t) + (1-t)^2 U_{m-1}^2(t) \qquad (-1 \leqq t \leqq 1), \tag{20}$$

where

$$T_m(t) = \cos m\theta \qquad (t = \cos\theta),$$

$$U_m(t) = \frac{1}{m+1} T'_{m+1}(t) = \frac{\sin(m+1)\theta}{\sin\theta}$$

are the Tchebycheff polynomials of the first and second kind respectively. In the notation of Theorem A, equation (20) expresses the polynomial $u(t)\equiv 1$ as a sum of the extreme polynomials $\underline{u}(t)=T_m^2(t)$ and $\bar{u}(t)=(1-t^2)U_{m-1}^2(t)$.

(i) Let $q(t)\equiv 0$ and $p(t)=1$ in Theorem 1. We then recognize the class of polynomials $\mathscr{U}$ defined in (4) for $n=2m$ as those polynomials $P_{2m}(t)=\sum_{i=0}^{n} a_i t^i$ satisfying

(21) $$0\leqq P_{2m}(t)\leqq 1,\quad -1\leqq t\leqq 1 \quad\text{and}\quad P_{2m}(-1)=0.$$

Invoking Theorem 1 we conclude that

$$P'_{2m}(-1)\leqq \frac{d}{dt}(1-t^2)U_{m-1}^2(t)|_{t=1}$$

with equality prevailing only if $P_{2m}(t)=v_*(t)=(1-t^2)U_{m-1}^2(t)$.

(ii) Problem 1 posed at the start of the paper was concerned with the task of calculating the maximum of

$$\max_{-1\leqq t\leqq 1} |P_{m-1}(t)|$$

over the set of polynomials of degree $m-1$ satisfying the condition

$$(1-t^2)P_{m-1}^2(t)\leqq 1 \qquad (-1\leqq t\leqq 1).$$

The solution to this problem is contained in the following slightly more general result. We consider the class $\mathscr{U}_0$ (see (6)) of polynomials satisfying (21) and the further condition that $P_{2m}(+1)=0$. Consulting Theorem 2 we may conclude that for $P_{2m}\in\mathscr{U}_0$ we have

(22) $$\max_{-1\leqq t\leqq 1}\frac{P_{2m}(t)}{1-t^2}\leqq \max_{-1\leqq t\leqq 1} U_{m-1}^2(t)=m^2$$

with equality present only when $P_{2m}(t)=(1-t^2)U_{m-1}^2(t)$.

Furthermore an induction argument using the relation

$$U_{m-1}(t)=\frac{\sin m\theta}{\sin\theta}=\cos(m-1)\theta+\cos\theta\,\frac{\sin(m-1)\theta}{\sin\theta} \qquad (m\geqq 1)$$

shows that equality on the right side of (22) is attained exclusively at $t=\pm 1$.

The solution of Problem 1 which we have just described may be cast in the following form: if $P_{m-1}(t)$ is a polynomial of degree $m-1$ on $[-1,1]$ and

$$\sqrt{1-t^2}\,|P_{m-1}(t)|\leqq 1$$

then

$$|P_{m-1}(t)|\leqq m$$

with equality only if $P_{m-1}(t)=\gamma U_{m-1}(t)$, ($|\gamma|=1$) and $t=\pm 1$.

(iii) In continuing to apply the results of Section 2 we consider Theorem 3. The polynomial $(1-t^2)U_{m-1}^2(t)$ vanishes at

$$t_k=-\cos\frac{k\pi}{m} \qquad (k=0,1,\dots,m)$$

and equals one at the zeros of $T_m(t)$, i.e., at

$$s_k = \cos\frac{2k-1}{2m}\pi, \qquad (k = 1, 2, \ldots, m).$$

Theorem 3 asserts that for $z \in (-1, s_1) \cup (t_1, s_2) \cup \ldots \cup (t_{m-1}, s_m)$ the polynomial $(1-t^2)U_{m-1}^2(t)$ has the maximum derivative at the point z among all polynomials P_{2m} obeying the contraints $0 \leqq P_{2m}(t) \leqq 1$ and $P_{2m}(z) = (1-z^2)U_{m-1}^2(z)$.

Example 4. The applications of the preceding paragraph involved the specific functions $p(t) \equiv 1$ and $q(t) \equiv 0$. Other specifications lead to new inequalities. For example we suppose again that $q(t) \equiv 0$ and now define $p(t)$ by

$$m(m+\alpha+\beta+1)p(t) = m(m+\alpha+\beta+1)\{P_m^{(\alpha,\beta)}(t)\}^2 + (1-t^2)\left\{\frac{d}{dt}P_m^{(\alpha,\beta)}(t)\right\}^2 \tag{23}$$

where $P_m^{(\alpha,\beta)}(t)$ are the Jacobi polynomials, orthogonal on $[-1, 1]$ with respect to the weight function $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\beta > -1, \alpha > -1)$ and normalized by the condition

$$P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}.$$

In order to ascertain the monotonicity properties of $p(t)$ required in Theorem 2 we proceed as follows. It is familiar that $y = P_m^{(\alpha,\beta)}(t)$ satisfies the differential equation

$$(1-t^2)y'' + [\beta - \alpha - (\alpha+\beta+2)t]y' + m(m+\alpha+\beta+1)y = 0$$

and therefore

$$m(m+\alpha+\beta+1)p'(t) = 2(\alpha - \beta + (\alpha+\beta+1)t)\left\{\frac{d}{dt}P_m^{(\alpha,\beta)}(t)\right\}^2 \tag{24}$$

(see Szegő [17, Chap. 7]) from which it is clear that $p'(t)$ changes sign once at $t_0 = (\beta-\alpha)/(\alpha+\beta+1)$. Note that $-1 < t_0 < 1$ if and only if $(\alpha+1/2)(\beta+1/2) > 0$.

If $\alpha = \beta = \lambda - 1/2$, the ultraspherical polynomials $P_m^{(\lambda)}(t)$ are defined by

$$P_m^{(\lambda)}(t) = \frac{\Gamma(\lambda+1/2)\Gamma(m+2\lambda)}{\Gamma(2\lambda)\Gamma(m+\lambda+1/2)} P_m^{(\lambda-1/2,\lambda-1/2)}(t), \qquad \lambda > -\frac{1}{2}.$$

In this case the calculation in (24) reduces to

$$m(m+2\lambda)p'(t) = 2\lambda t\left\{\frac{d}{dx}P_m^{(\alpha,\beta)}(t)\right\}^2.$$

If $\lambda > 0$ then $p'(t) > 0$ for $0 < t < 1$ and $p'(t) < 0$ for $-1 < t < 0$. Therefore

$$f(t) = \frac{p(t)}{1-t^2}$$

satisfies the monotonicity conditions stipulated in Theorem 2.

By applying Theorem 2 with the function

$$m(m+2\lambda)p(t) = m(m+2\lambda)\{P_m^{(\lambda)}(t)\}^2 + (1-t^2)\left\{\frac{d}{dt}P_m^{(\lambda)}(t)\right\}^2$$

we obtain the result that any polynomial $Q_{2m}(t)$ of degree $\leqq 2m$ obeying the restrictions $0 \leqq Q_{2m}(t) \leqq m(m+2\lambda)p(t)$ and $Q_{2m}(\pm 1)=0$ also fulfills the inequality

$$\max_{-1 \leqq t \leqq 1} \frac{Q_{2m}(t)}{1-t^2} \leqq \max_{-1 \leqq t \leqq 1} \left\{ \frac{d}{dt} P_m^{(\lambda)}(t) \right\}^2 \tag{25}$$

and equality occurs only if $Q_{2m}(t) = (1-t^2)\left\{ \frac{d}{dt} P_m^{(\lambda)}(t) \right\}^2$.

For $p(t)$ defined by (23) Theorems 1 and 3 apply for all values $\alpha > -1$ and $\beta > -1$.

These examples are typical expressions of Theorems $1-7$. We refer the reader to [11] for other applications and refinements of these ideas.

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On certain classes of power-bounded operators in Hilbert space

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Dedicated to P. R. Halmos on his 50th birthday

1. Let $\mathscr{C}_\varrho$ $(\varrho > 0)$ denote the class of those (bounded, linear) operators T in Hilbert space $\mathfrak{H}$, whose powers T^n $(n=1, 2, \ldots)$ admit a representation

$$T^n = \varrho \cdot \mathrm{pr}\, U^n \qquad (n=1, 2, \ldots), \tag{1}$$

where U is a unitary operator in some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ as a subspace.

It is known that the class $\mathscr{C}_1$ consists precisely of the contraction operators T, i.e. for which

$$\|T\| \leqq 1, \tag{2}$$

cf. [1], and that $\mathscr{C}_2$ consists precisely of those T for which

$$w(T) \leqq 1. \tag{3}$$

The latter fact was discovered by C. A. BERGER (not yet published); simplified proofs appear in [2] and [3]. *Norm* $\|T\|$ and *numerical radius* $w(T)$ of an operator are defined by

$$T = \sup \frac{\|Th\|}{\|h\|}, \quad w(T) = \sup \frac{|(Th, h)|}{\|h\|^2} \qquad (h \in \mathfrak{H}, h \neq 0).$$

Clearly, every operator T of class $\mathscr{C}_\varrho$ is power-bouded, indeed we have $\|T^n\| \leqq \varrho$, but the converse is not true. We shall give an example of a power-bounded operator which is not contained in any of the classes $\mathscr{C}_\varrho (\varrho > 0)$.

2. First we give a characterization of the classes $\mathscr{C}_\varrho$.

Theorem. *In order that the operator T in $\mathfrak{H}$ belong to the class $\mathscr{C}_\varrho$ it is necessary and sufficient that the following conditions be satisfied:*

$$\|h\|^2 - 2\left(1-\frac{1}{\varrho}\right) \mathrm{Re}\,(zTh, h) + \left(1-\frac{2}{\varrho}\right) \|zTh\|^2 \geqq 0 \quad \textit{for} \quad h \in \mathfrak{H} \quad \textit{and} \quad |z| \leqq 1, \tag{I_ϱ}$$

(II) *the spectrum of T lies in the closed unit disk.*

For $\varrho \leqq 2$ condition (I_ϱ) *implies* (II).

Proof. Suppose that (1) holds. Since U is unitary, the series $I_{\mathfrak{K}}+2zU+\cdots+$ $+2z^nU^n+\cdots$ converges in the norm for every z, $|z|<1$, its sum being equal to $(I_{\mathfrak{K}}+zU)(I_{\mathfrak{K}}-zU)^{-1}$. By virtue of (1), the series $I_{\mathfrak{H}}+\frac{2}{\varrho}zT+\cdots+\frac{2}{\varrho}z^nT^n+\cdots$ also will converge in the norm, its sum being then equal to $\left(1-\frac{2}{\varrho}\right)I_{\mathfrak{H}}+\frac{2}{\varrho}(I_{\mathfrak{H}}-zT)^{-1}$. Hence we infer first that $(\mu I_{\mathfrak{H}}-T)^{-1}$ exists as a bounded operator in $\mathfrak{H}$ for $|\mu|>1$, i.e. condition (II) is fulfilled. Moreover, we have

$$\left(1-\frac{2}{\varrho}\right)I+\frac{2}{\varrho}(I-zT)^{-1}=\mathrm{pr}\,(I+zU)(I-zU)^{-1}\quad(|z|<1). \tag{4}$$

Since

$$\mathrm{Re}\big((I+zU)k,\,(I-zU)k\big)=\|k\|^2-|z|^2\|Uk\|^2=(1-|z|^2)\|k\|^2\geqq 0\ \text{ for } k\in\mathfrak{K},\ |z|<1,$$

we have

$$\mathrm{Re}\big((I+zU)(I-zU)^{-1}k,\,k\big)\geqq 0\quad\text{for}\quad k\in\mathfrak{K},\ |z|<1,$$

thus, by (4),

$$\mathrm{Re}\left[\left(1-\frac{2}{\varrho}\right)(l,l)+\frac{2}{\varrho}\big((I-zT)^{-1}l,l\big)\right]\geqq 0\quad\text{for}\quad l\in\mathfrak{H},\ |z|<1. \tag{5}$$

Set $l=l_z=(I-zT)h$, where h is an arbitrary element of $\mathfrak{H}$. Then (5) yields

$$\left(1-\frac{2}{\varrho}\right)\|(I-zT)h\|^2+\frac{2}{\varrho}\,\mathrm{Re}\big(h,(I-zT)h\big)\geqq 0\quad\text{for}\quad h\in\mathfrak{H},\ |z|<1, \tag{6}$$

whence (I_ϱ) follows by a simple rearrangement, at least for $|z|<1$. The limit case $|z|=1$ can be included by continuity.

Suppose now, conversely, that (I_ϱ) and (II) hold for T. By (II), $(I-zT)^{-1}$ exists as a bounded operator in $\mathfrak{H}$, for $|z|<1$. From (I_ϱ) we obtain (6) by the inverse of the above mentioned rearrangement. Setting $h=h_z=(I-zT)^{-1}l$ in (6), where l is an arbitrary element of $\mathfrak{H}$, we get (5). This means that the operator valued function

$$F(z)=\left(1-\frac{2}{\varrho}\right)I+\frac{2}{\varrho}(I-zT)^{-1} \tag{7}$$

satisfies the condition

$$\mathrm{Re}\,F(z)\geqq O. \tag{8}$$

Since, moreover, $F(z)$ is holomorphic in the unit disc ($|z|<1$), and $F(0)=I$, it follows from a theorem of F. Riesz, generalized to operator valued functions, that there exists a unitary operator U in some space $\mathfrak{K}(\supseteq\mathfrak{H})$, such that

$$F(z)=\mathrm{pr}\,(I+zU)(I-zU)^{-1}\quad(|z|<1), \tag{9}$$

cf. e.g. [1]. Since

$$(I+zU)(I-zU)^{-1}=I+2zU+\cdots+2z^nU^n+\cdots\quad\text{for}\quad|z|<1,$$

and, by (7),

$$F(z)=I+\frac{2}{\varrho}zT+\cdots+\frac{2}{\varrho}z^nT^n+\cdots\quad\text{at least for}\ |z|\,\|T\|<1,$$

it results from (9) by comparing coefficients that

$$\frac{1}{\varrho} T^n = \mathrm{pr}\, U^n \qquad (n = 1, 2, \ldots),$$

i.e. $T \in \mathscr{C}_\varrho$.

Thus we have proved that (I_ϱ) and (II) characterize the operators of class $\mathscr{C}_\varrho$.

We have still to prove the last statement of the theorem. We start with relation (6) which is an equivalent form of (I_ϱ). If $\varrho \leqq 2$ we have $1-2/\varrho \leqq 0$ so that (6) implies

$$\mathrm{Re}\big(h, (I-zT)h\big) \geqq 0 \qquad (h \in \mathfrak{H},\ |z| \leqq 1);$$

choosing an adequate value for z we obtain hence

$$\|h\|^2 \geqq |(Th, h)| \qquad (h \in \mathfrak{H}). \tag{10}$$

Consider the self-adjoint operator $R_z = \mathrm{Re}\,(I-zT)$. Since

$$(R_z h, h) = \mathrm{Re}\,\big((I-zT)h, h\big) = \|h\|^2 - \mathrm{Re}\, z(Th, h) \geqq (1-|z|)\|h\|^2,$$

we have $R_z \geqq (1-|z|)I$, thus if $|z|<1$ then $Q_z = R_z^{1/2} \geqq (1-|z|)^{1/2} I$, Q_z^{-1} exists as a bounded, everywhere defined operator, $\|Q_z^{-1}\| \leqq (1-|z|)^{-1/2}$. We have for $|z|<1$

$$I-zT = R_z + i\,\mathrm{Im}\,(I-zT) = R_z - i\,\mathrm{Im}\,(zT) = Q_z[I - iQ_z^{-1}\,\mathrm{Im}\,(zT)Q_z^{-1}]Q_z.$$

Since $Q_z^{-1}\,\mathrm{Im}\,(zT)\,Q_z^{-1}$ is selfadjoint, the operator in [ ] has an inverse, everywhere defined and bounded by 1. Thus $I-zT$ also has a bounded and everywhere defined inverse; indeed,

$$\|(I-zT)^{-1}\| \leqq (1-|z|)^{-1} \qquad (|z|<1).$$

This implies (II), moreover the inequality

$$\|(\mu I - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \quad \text{for} \quad 1 < |\mu| < \infty. \tag{10'}$$

This concludes the proof of the theorem.

It is clear that for $\varrho = 1$ and $\varrho = 2$, (I_ϱ) reduces to condition (2) and (3), respectively. Thus our theorem generalizes the characterizations of the classes $\mathscr{C}_1$ and $\mathscr{C}_2$ mentioned in § 1.

We may complete this remark with the following ones:

Remark 1. *If* $0<\varrho<2$, $\varrho \neq 1$, (I_ϱ) *reduces to the condition*

$$\|(\mu I - T)h\| \leqq \frac{|\mu|}{|\varrho - 1|}\|h\| \quad \textit{for} \quad \left|\frac{\varrho-1}{\varrho-2}\right| \leqq |\mu| < \infty, \quad h \in \mathfrak{H}, \tag{I'_ϱ}$$

while if $2<\varrho<\infty$, (I_ϱ) *reduces to the condition*

$$\|(\mu I - T)h\| \geqq \frac{|\mu|}{\varrho - 1}\|h\| \quad \textit{for} \quad \frac{\varrho-1}{\varrho-2} \leqq |\mu| < \infty, \quad h \in \mathfrak{H}. \tag{I''_ϱ}$$

Proof. If $0<\varrho<2$, $\varrho\neq 1$, multiplication by the negative factor $\varrho/(\varrho-2)$, and an easy rearrangement transforms (I_ϱ) into the equivalent form

$$\left\|\left(\frac{\varrho-1}{\varrho-2}I-zT\right)h\right\|^2-\frac{1}{(\varrho-2)^2}\|h\|^2\leqq 0 \qquad (h\in\mathfrak{H},\ |z|\leqq 1). \tag{11}$$

Setting $z=\dfrac{\varrho-1}{\varrho-2}\dfrac{1}{\mu}$, (11) can be expressed in the equivalent form (I'_ϱ).

If $2<\varrho<\infty$, multiplication by the positive factor $\varrho/(\varrho-2)$ and the same easy rearrangement transforms (I_ϱ) into the equivalent form

$$\left\|\left(\frac{\varrho-1}{\varrho-2}I-zT\right)h\right\|^2-\frac{1}{(\varrho-2)^2}\|h\|^2\geqq 0 \qquad (h\in\mathfrak{H},\ |z|\leqq 1). \tag{12}$$

Setting, as above, $z=\dfrac{\varrho-1}{\varrho-2}\dfrac{1}{\mu}$, (12) transforms into the equivalent form (I''_ϱ).

Remark 2. *In order that T be of class $\mathscr{C}_\varrho$ with $1<\varrho<2$, it is necessary and sufficient that the condition*

$$(\mathrm{III}'_\varrho) \qquad \|\mu I-T\|\leqq|\mu|+1 \quad \textit{for} \quad \frac{\varrho-1}{2-\varrho}\leqq|\mu|<\infty$$

hold.

Proof. (III'_ϱ) implies (I'_ϱ) since

$$|\mu|+1\leqq\frac{|\mu|}{\varrho-1}$$

for $|\mu|\geqq(\varrho-1)(2-\varrho)^{-1}$. On the other hand, if $|\mu|\geqq(\varrho-1)(2-\varrho)^{-1}$ and $\mu=\varepsilon|\mu|$, (I'_ϱ) gives

$$\|\mu I-T\|\leqq\left|\mu-\varepsilon\frac{\varrho-1}{2-\varrho}\right|+\left\|\varepsilon\frac{\varrho-1}{2-\varrho}I-T\right\|\leqq|\mu|-\frac{\varrho-1}{2-\varrho}+\frac{1}{2-\varrho}=|\mu|+1,$$

thus (I'_ϱ) implies (III'_ϱ).

Remark 3. *In order that T be of class $\mathscr{C}_\varrho$ with $2\leqq\varrho<\infty$, it is necessary and sufficient that T verify the conditions* (II) *and*

$$(\mathrm{III}''_\varrho) \qquad \|(\mu I-T)^{-1}\|\leqq\frac{1}{|\mu|-1}\begin{cases}\textit{for}\ \ 1<|\mu|<\infty & \textit{if}\ \ \varrho=2,\\ \textit{for}\ \ 1<|\mu|\leqq r_\varrho=\dfrac{\varrho-1}{\varrho-2} & \textit{if}\ \ \varrho>2.\end{cases}$$

Proof. *Case* $\varrho=2$. We know already that (I_2) implies (10′), i.e. (III''_2). Suppose, conversely, that (III''_2) holds. Then $\|(\mu I-T)h\|\geqq(|\mu|-1)\|h\|$ for $1<|\mu|<\infty$ hence $\|(I-\varepsilon rT)h\|\geqq(1-r)\|h\|$ for $0<r<1$, $|\varepsilon|=1$. This gives

$$0\leqq\|h-\varepsilon rTh\|^2-(1-r)^2\|h\|^2=2r\|h\|^2-2\,\mathrm{Re}\,\varepsilon r(Th,h)+r^2\|Th\|^2-r^2\|h\|^2.$$

Dividing by $2r$ and letting $r\to 0$ it results $\|h\|^2-\mathrm{Re}\,\varepsilon(Th,h)\geqq 0$. Since this holds for arbitrary ε, $|\varepsilon|=1$, we get $|(Th,h)|\leqq\|h\|^2$ for any $h\in\mathfrak{H}$, i.e. $w(T)\leqq 1$. Thus (III''_2) implies (3), i.e. (I_2).

Case $\varrho>2$. Suppose first that (I_ϱ), i.e. (I''_ϱ) holds. Then we have $\|(\mu I-T)h\|\geqq \geqq \frac{|\mu|}{\varrho-1}\|h\|$ for $|\mu|\geqq r_\varrho$, in particular

$$\|(\varepsilon r_\varrho I-T)h\|\geqq\frac{r_\varrho}{\varrho-1}\|h\|=\frac{1}{\varrho-2}\|h\| \qquad (|\varepsilon|=1, h\in\mathfrak{H}).$$

If $1<|\mu|<r_\varrho$ and $\varepsilon=\mu/|\mu|$, we obtain hence

$$\|(\mu I-T)h\|\geqq\|(\varepsilon r_\varrho I-T)h\|-\|(\varepsilon r_\varrho-\mu)h\|\geqq\left[\frac{1}{\varrho-2}-(r_\varrho-|\mu|)\right]\|h\|,$$

i. e.

$$\|(\mu I-T)h\|\geqq(|\mu|-1)\|h\| \quad \text{for} \quad 1<\mu<r_\varrho.$$

This implies (III''_ϱ).

Suppose, conversely, that (III''_ϱ) holds. Then we have in particular

$$(13) \quad \|(I-\zeta T)^{-1}\|=\frac{1}{|\zeta|}\left\|\left(\frac{1}{\zeta}I-T\right)^{-1}\right\|\leqq r_\varrho\frac{1}{r_\varrho-1}=\varrho-1 \quad \text{for} \quad |\zeta|=\frac{1}{r_\varrho}.$$

Since, by (II), $(I-zT)^{-1}$ is a holomorphic function of z for $|z|<1$, we conclude from (13) by the maximum principle that

$$\|(I-zT)^{-1}\|\leqq\varrho-1 \quad \text{for} \quad |z|\leqq\frac{1}{r_\varrho}.$$

Thus, if $|\mu|\geqq r_\varrho$, we have

$$|\mu|\cdot\|(\mu I-T)^{-1}\|=\left\|\left(I-\frac{1}{\mu}T\right)^{-1}\right\|\leqq\varrho-1,$$

i.e. (I''_ϱ). This finishes the proof.

3. Let $\|T\|\leqq 1$. Then for every complex μ we have $\|\mu I-T\|\leqq|\mu|+\|T\|\leqq|\mu|+1$; thus in virtue of Remark 2 we have

$$(14) \qquad \mathscr{C}_1\subset\mathscr{C}_\varrho \quad \text{for } 1\leqq\varrho<2.$$

Let now $T\in\mathscr{C}_{\varrho_1}$ with $0<\varrho_1<\infty$, and let ϱ_2 be such that $\varrho_1\leqq\varrho_2<2\varrho_1$. Since $T\in\mathscr{C}_{\varrho_1}$, there exists a unitary operator U in some Hilbert space $\mathfrak{K}\supseteq\mathfrak{H}$ such that

$$(15) \qquad T^n=\varrho_1\cdot\mathrm{pr}\,U^n \qquad (n=1, 2, \ldots).$$

Since $U\in\mathscr{C}_1$ and $1\leqq\varrho_2/\varrho_1<2$, we have $U\in\mathscr{C}_{\varrho_2/\varrho_1}$, by (14). Thus there exists a unitary operator V in a Hilbert space $\mathfrak{L}\supseteq\mathfrak{H}$ such that

$$(15') \qquad U^n=\frac{\varrho_2}{\varrho_1}\mathrm{pr}\,V^n \qquad (n=1, 2, \ldots).$$

Comparing (15) with (15′) we obtain $T^n=\varrho_2\,\mathrm{pr}\,V^n$ $(n=1, 2, \ldots,)$, i.e. $T\in\mathscr{C}_{\varrho_2}$. From this remark it follows readily the following

Proposition 1. *The classes* $\mathscr{C}_\varrho(0<\varrho<\infty)$ *form a non-decreasing scale, i.e.*

$$(16) \qquad \mathscr{C}_{\varrho_1}\subset\mathscr{C}_{\varrho_2} \quad \text{if} \quad 0<\varrho_1<\varrho_2<\infty.$$

In order to complete this result let us consider a simple example. Let T_s $(s>0)$ be the operator in complex Euclidean 2-space with the matrix $\begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$. Obviously, $\|T_s\|=s$, $T_s^2=O$, and

$$(\mu I-T_s)^{-1} = \frac{1}{\mu^2}(\mu I+T_s).$$

Hence the spectrum of T_s consists of the single point 0, thus condition (II) is fulfilled. Moreover, we have

$$\|(\mu I-T_s)^{-1}\| \leqq \frac{|\mu|+s}{|\mu|^2} = \left(1+\frac{s}{|\mu|}\right)\frac{|\mu|-1}{|\mu|}\frac{1}{|\mu|-1} \leqq \frac{1}{|\mu|-1}$$

if $1<|\mu|\leqq\frac{s}{s-1}$. Thus, if $s\geqq 1$, condition (III_ϱ'') also is fulfilled, with $\varrho=s+1$. Thus

$$T_s\in\mathscr{C}_{s+1} \quad \text{if} \quad s\geqq 1,$$

but, since $\|T_s\|=s$, T_s does not belong to any of the classes $\mathscr{C}_\varrho$ with $\varrho<s$.

This shows that the increasing scale of the classes $\mathscr{C}_\varrho$ does not attain a maximum (indeed, $\mathscr{C}_\varrho$ is properly contained in $\mathscr{C}_{\varrho'}$, if $1\leqq\varrho<\varrho'-1$).

Now, let $0<s<1$. Then, putting $\varrho=\frac{2s}{1+s}$, we have

$$\|\mu I-T_s\| \leqq |\mu|+s \leqq \frac{|\mu|}{1-\varrho} \quad \text{for} \quad \frac{1-\varrho}{2-\varrho} \leqq |\mu|<\infty,$$

i.e. (I_ϱ') is verified. Hence $T_s\in\mathscr{C}_\varrho$. Since $s=\frac{\varrho}{2-\varrho}$, this result also can be expressed in the form

$$T_{\frac{\varrho}{2-\varrho}}\in\mathscr{C}_\varrho \quad \text{if} \quad 0<\varrho<1.$$

But $\|T_s\|=s$ again implies that $T_{\frac{\varrho}{2-\varrho}}$ does not belong to any of the classes $\mathscr{C}_{\varrho'}$ with $\varrho'<\frac{\varrho}{2-\varrho}$.

This shows that none of the classes $\mathscr{C}_\varrho$ is void and, moreover, that the scale of the classes $\mathscr{C}_\varrho$ does not attain a minimum $\left(\text{indeed, } \mathscr{C}_\varrho \text{ properly contains } \mathscr{C}_{\varrho'} \text{ if } 1>\varrho>2\frac{\varrho'}{\varrho'+1}\right)$.

Thus none of the classes $\mathscr{C}_\varrho(\varrho>0)$ is void, and the scale of the classes $\mathscr{C}_\varrho$ neither attains a minimum nor a maximum.

4. There exist power-bouded operators which do not belong to any of the classes $\mathscr{C}_\varrho$. More precisely, *we shall give an example of an operator T such that $\|T^n\|\leqq 2$ for every integer n, and which is not contained in any of the classes $\mathscr{C}_\varrho(\varrho>0)$.*

Consider, to this effect, the space $L^2(-1, 1)$, and the operators V and A defined in this space by

$$(Vf)(x) = f(-x) \quad \text{and} \quad (Af)(x) = a(x)f(x),$$

where $a(x) = 2^{1/2}$ if $-1<x<0$, and $=2^{-1/2}$ if $0<x<1$.

Set $T=AVA^{-1}$, i.e.

$$(Tf)(x) = \frac{a(x)}{a(-x)} f(-x). \tag{17}$$

Since $V^2=I$, we have $T^2=I$, i.e. $T^n = \begin{cases} T \text{ for } n \text{ odd,} \\ I \text{ for } n \text{ even.} \end{cases}$ Let us show that $\|T\|=2$. Indeed, we have

$$\|Tf\|^2 = \int_{-1}^{+1} \left| \frac{a(x)}{a(-x)} f(-x) \right|^2 dx \leqq 4\|f\|^2 \quad \text{for every} \quad f\in L^2,$$

because
$$\frac{a(x)}{a(-x)} = \begin{cases} 2 & \text{if} \quad -1<x<0, \\ 1/2 & \text{if} \quad 0<x<1, \end{cases}$$

and
$$\|Tf\|^2=4\|f\|^2 \quad \text{if} \quad f(x)=0 \quad \text{for} \quad -1<x<0.$$

We assert that T belongs to none of the classes $\mathscr{C}_\varrho$.

Since $\|T\|=2$, the values $\varrho<2$ are *a priori* impossible. We shall show, using the condition (III$''_\varrho$), that the values $\varrho \geqq 2$ also are impossible.

To this end, observe first that, since $T^2=I$, we have

$$(\mu I - T)^{-1} = \frac{1}{\mu^2-1}(\mu I + T) \quad \text{for} \quad \mu \neq \pm 1. \tag{18}$$

Choose the function $f_0(x) = \begin{cases} 0 \text{ for } -1<x<0 \\ 1 \text{ for } 0<x<1 \end{cases}$. Then

$$\|(\mu I + T)f_0\|^2 = \int_{-1}^{1} \left| \mu f_0(x) + \frac{a(x)}{a(-x)} f_0(-x) \right|^2 dx =$$
$$= |\mu|^2+4 = (|\mu|^2+4)\|f_0\|^2,$$

whence
$$\|\mu I + T\| \geqq (|\mu|^2+4)^{1/2}$$

and by (18),
$$\|(\mu I - T)^{-1}\| \geqq \frac{1}{|\mu^2-1|}(|\mu|^2+4)^{1/2}.$$

Now, if μ is real, $\mu>1$, we have

$$\frac{1}{\mu^2-1}(\mu^2+4)^{1/2} > \frac{1}{\mu-1}$$

for μ sufficiently close to 1, namely for $1<\mu<1.5$, and this shows that condition (III''_ϱ) is not satisfied for any $\varrho \geqq 2$.

5. Denote by A the class of the functions

$$u(z) = \sum_0^\infty a_n z^n \quad \text{with} \quad \sum_0^\infty |a_n| < \infty.$$

From (1) it follows for every $T \in \mathscr{C}_\varrho$ and $u \in A$:

$$u(T) = \mathrm{pr}\,[\varrho \cdot u(U) + (1-\varrho)\cdot u(0) I_{\mathfrak{K}}]. \tag{19}$$

This relation implies, by virtue of the spectral theory for unitary operators, the following

Proposition 2. *For $T \in \mathscr{C}_\varrho$ and $u \in A$ we have*

$$\|u(T)\| \leqq \max_{|z|\leqq 1} |u_\varrho(z)| \tag{20}$$

and

$$\big[\min_{|z|\leqq 1} \mathrm{Re}\, u_\varrho(z)\big] I_{\mathfrak{H}} \leqq \mathrm{Re}\, u(T) \leqq \big[\max_{|z|\leqq 1} \mathrm{Re}\, u_\varrho(z)\big] I_{\mathfrak{H}} \tag{21}$$

where

$$u_\varrho(z) = \varrho \cdot u(z) + (1-\varrho) \cdot u(0).$$

Obviously, (20) and (21) generalize, for the classes $\mathscr{C}_\varrho$, the inequalities of von Neumann and Heinz, respectively, on contractions, i.e. for the class $\mathscr{C}_1$. *Cf.* [4], p. 431.

It is clear that if $\varrho=1$, (20) implies $\|T\| \leqq 1$: one has only to set $u(z)=z$. In the case $\varrho \neq 1$, (20) does not seem to imply that $T \in \mathscr{C}_\varrho$. But (21) does: in fact we shall prove the following

Proposition 3. *Suppose T is a power-bounded operator which satisfies* (21) *for every function $u \in A$. Then $T \in \mathscr{C}_\varrho$.*

Proof. Since T is power-bounded, its spectrum is contained in the unit disc, i.e. T satisfies (II). Moreover, power-boundedness assures that $u(T) = a_0 I + a_1 T + \cdots + a_n T^n + \cdots$ converges in norm. Concerning (I_ϱ) it suffices to verify (5), or, equivalently, (8). To this effect, choose

$$u(z) = u(\zeta; z) = 1 - \frac{2}{\varrho} + \frac{2}{\varrho}\frac{1}{1-\zeta z} = 1 + \frac{2}{\varrho}\zeta z + \frac{2}{\varrho}\zeta^2 z^2 + \cdots \qquad (|\zeta|<1,\ |z|\leqq 1).$$

Then
$$u_\varrho(z) = 1 + 2\zeta z + 2\zeta^2 z^2 + \cdots = \frac{1+\zeta z}{1-\zeta z},$$

hence $\mathrm{Re}\, u_\varrho(z) \geqq 0$ for $|z| \leqq 1$. Thus, by (21),

$$O \leqq \mathrm{Re}\, u(T) = \left(1 - \frac{2}{\varrho}\right) I_{\mathfrak{H}} + \frac{2}{\varrho}(I - \zeta T)^{-1},$$

and this result coincides with (8).

Finally, we make the following

Proposition 4. *Let* $u(z)\in A$, *with* $|u(z)|\leqq 1$ *for* $|z|\leqq 1$, *and* $u(0)=0$. *Then* $T\in\mathscr{C}_\varrho$ *implies* $u(T)\in\mathscr{C}_\varrho$.

Proof. Since $u_n(z)=[u(z)]^n$ also belongs to A for every $n=1, 2, \ldots$, and since $u_n(0)=0$, we have for $T\in\mathscr{C}_\varrho$ by (19):

$$u(T)^n=u_n(T)=\mathrm{pr}\ \varrho\cdot u_n(U)=\varrho\cdot\mathrm{pr}\ u(U)^n. \tag{22}$$

Now, since $|u(z)|\leqq 1$ for $|z|\leqq 1$, $u(U)$ is a contraction. Thus there exists a unitary operator V such that $u(U)^n=\mathrm{pr}\ V^n$ $(n=0, 1, \ldots)$. Comparing this with (22) we conclude that

$$u(T)^n=\varrho\cdot\mathrm{pr}\ V^n \qquad (n=1, 2, \ldots),$$

i.e. $u(T)\in\mathscr{C}_\varrho$.

For $\varrho=2$, Proposition 4 reduces to a result obtained by STAMPFLI, *cf.* [2].

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Sur les contractions de l'espace de Hilbert. XII
Fonctions intérieures admettant des facteurs extérieurs

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Soit T une contraction complètement non-unitaire de l'espace de Hilbert $\mathfrak{H}$ et soit $\Theta_T(\lambda)$ sa fonction caractéristique (*cf.* [1]). On sait qu'il y a une correspondance biunivoque entre les sous-espaces de $\mathfrak{H}$ (non banaux), invariants pour T, et les factorisations de $\Theta_T(\lambda)$ en produit de deux fonctions analytiques contractives (non constantes unitaires)[1]), vérifiant encore une certaine condition supplémentaire (*cf.* [2], § 3). Par exemple, si $\Theta_T(\lambda)$ est intérieure (c'est le cas si $T^{*n} \to O$), cette condition supplémentaire exige que les facteurs soient aussi des fonctions intérieures (*cf.* [2], § 4).

Cela impose le problème de savoir s'il existe du tout des factorisations de $\Theta_T(\lambda)$ en produit de deux fonctions analytiques contractives, non constantes unitaires (satisfaisant à la condition supplémentaire mentionnée, ou non). Une réponse négative donnerait immédiatement un exemple d'un opérateur n'admettant pas de sous-espaces invariants non banaux.

Nous verrons que cette situation (heureuse?) ne se présente pas. En effet, nous allons démontrer que *toute* fonction analytique contractive (non constante unitaire) admet des factorisations non banales (Théorème 2). Le point principal dans cette démonstration est de construire, pour toute fonction *intérieure* non constante de type $\{\mathfrak{E}, \mathfrak{E}, \Theta(\lambda)\}$, où $\dim \mathfrak{E} = \infty$, une factorisation en produit d'une fonction *extérieure* et d'une fonction *intérieure*, aucune d'elles n'étant une constante unitaire (Théorème 1).

1. Rappelons, pour commencer, que par une *translation unilatérale* dans un espace $\mathfrak{H}$ de Hilbert on entend une isométrie V dans $\mathfrak{H}$ telle que

$$\mathfrak{H} = \bigoplus_{n=0}^{\infty} V^n \mathfrak{A} \quad \text{où} \quad \mathfrak{A} = \mathfrak{H} \ominus V\mathfrak{H},$$

la *multiplicité* de la translation unilatérale V étant égale, par définition, à $\dim \mathfrak{A}$. La projection orthogonale de $\mathfrak{H}$ sur $\mathfrak{A}$ est fournie par

$$P_{\mathfrak{A}} = I - VV^*.$$

V étant une translation unilatérale dans $\mathfrak{H}$, appelons $\pi(V)$ la classe des opérateurs (linéaires bornés) Q de $\mathfrak{H}$ auxquels on peut associer un espace de Hilbert $\mathfrak{H}_Q$,

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[1]) On dira que la fonction $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ est constante unitaire si $\Theta(\lambda) \equiv \Theta_0$ ($|\lambda| < 1$) où Θ_0 est une transformation unitaire de $\mathfrak{E}$ sur $\mathfrak{E}_*$.

une translation unilatérale V_Q dans $\mathfrak{H}_Q$, et un opérateur A de $\mathfrak{H}_Q$ dans $\mathfrak{H}$, tels que

$$VA = AV_Q \tag{1}$$

et

$$AA^* = Q. \tag{2}$$

Proposition 1. *Pour qu'un opérateur autoadjoint Q de $\mathfrak{H}$, $Q \geqq O$, appartienne à la classe $\pi(V)$, il faut et il suffit qu'on ait*

$$Q - VQV^* \geqq O. \tag{3}$$

Démonstration. Lorsque $Q \in \pi(V)$ il s'ensuit de (1) et (2):

$$Q - VQV^* = AA^* - VAA^*V^* = AA^* - AV_QV_Q^*A^* = A(I - V_QV_Q^*)A^* \geqq O$$

puisque $I - V_QV_Q^* \geqq O$.

Supposons, inversement, que Q vérifie (3). Désignons par R la racine carrée positive de l'opérateur au premier membre de (3). On a alors $Q = R^2 + VQV^*$, d'où il s'ensuit par itération

$$Q = R^2 + VR^2V^* + \cdots + V^nR^2V^{*n} + V^{n+1}QV^{*n+1} \qquad (n = 1, 2, \ldots).$$

Or, comme V est une translation unilatérale, V^{*n} tend vers O lorsque n croît indéfiniment. Donc il résulte

$$Q = \sum_{n=0}^{\infty} V^nR^2V^{*n}$$

et par conséquent

$$\|Q^{1/2}h\|^2 = \sum_{n=0}^{\infty} \|RV^{*n}h\|^2 \quad \text{pour tout} \quad h \in \mathfrak{H}. \tag{4}$$

Envisageons l'espace de Hilbert $\mathfrak{H}_Q$ des suites $\mathbf{x} = \{x_n\}_0^\infty$ telles que $x_n \in \overline{R\mathfrak{H}}$ $(n = 0, 1, \ldots)$ et $\|\mathbf{x}\|^2 = \sum_0^\infty \|x_n\|^2 < \infty$. Soit V_Q la translation unilatérale dans $\mathfrak{H}_Q$, définie par

$$V_Q\{x_0, x_1, x_2, \ldots\} = \{0, x_0, x_1, \ldots\}.$$

Posons, pour $h \in \mathfrak{H}$,

$$Bh = \{Rh, RV^*h, RV^{*2}h, \ldots\}.$$

Il ressort de (4) que B est un opérateur de $\mathfrak{H}$ dans $\mathfrak{H}_Q$ tel que

$$\|Bh\|_{\mathfrak{H}_Q} = \|Q^{1/2}h\|_{\mathfrak{H}}. \tag{5}$$

De plus, comme

$$V_Q^*\{x_0, x_1, \ldots\} = \{x_1, x_2, \ldots\},$$

on a

$$BV^*h = \{RV^*h, RV^{*2}h, \ldots\} = V_Q^*Bh,$$

donc $BV^* = V_Q^*B$, $VB^* = B^*V_Q$. L'opérateur $A = B^*$ vérifie donc (1) et, grâce à (5), aussi (2).

Proposition 2. *Soit Q un opérateur de $\mathfrak{H}$, de classe $\pi(V)$ et tel que $O \leqq Q \leqq I$. L'opérateur $Q_\alpha = \alpha Q + (1-\alpha)I$, où $0 < \alpha < 1$, appartient alors aussi à la classe $\pi(V)$. Dans le cas où V est de multiplicité infinie, on peut choisir $\mathfrak{H}_{Q_\alpha} = \mathfrak{H}$ et $V_{Q_\alpha} = V$, donc, dans ce cas, il existe un opérateur A_α de $\mathfrak{H}$ tel que A_α permute à V et que $Q_\alpha = A_\alpha A_\alpha^*$.*

Démonstration. La première assertion résulte immédiatement de la proposition 1. Quant à la seconde assertion, observons d'abord que

$$R_\alpha^2 = Q_\alpha - VQ_\alpha V^* = \alpha(Q - VQV^*) + (1-\alpha)(I - VV^*) \geqq (1-\alpha)P_{\mathfrak{A}},$$

d'où il dérive $\overline{R_\alpha \mathfrak{H}} \supset (1-\alpha)P_{\mathfrak{A}}\mathfrak{H} = P_{\mathfrak{A}}\mathfrak{H} = \mathfrak{A}$ et par conséquent

$$\dim \mathfrak{H} \geqq \dim \overline{R_\alpha \mathfrak{H}} \geqq \dim \mathfrak{A}. \tag{6}$$

D'autre part, on a

$$\dim \mathfrak{H} = \aleph_0 \cdot \dim \mathfrak{A} = \dim \mathfrak{A}, \tag{7}$$

puisque $\dim \mathfrak{A}$ est infinie. (6) et (7) entraînent que $\dim \overline{R_\alpha \mathfrak{H}} = \dim \mathfrak{A}$. Ainsi, il existe une application unitaire φ de $\overline{R_\alpha \mathfrak{H}}$ sur $\mathfrak{A}$. Celle-ci induit une application unitaire Φ de l'espace $\mathfrak{H}_{Q_\alpha}$ des suites $\mathbf{x} = \{x_n\}_n^\infty$ $(x_n \in \overline{R_\alpha \mathfrak{H}})$ sur l'espace $\mathfrak{H}$, notamment en posant

$$\Phi \mathbf{x} = \sum_{n=0}^{\infty} V^n(\varphi x_n).$$

On a

$$\Phi(V_{Q_\alpha}\mathbf{x}) = \sum_{n=1}^{\infty} V^n(\varphi x_{n-1}) = V \sum_{n=1}^{\infty} V^{n-1}(\varphi x_{n-1}) = V\Phi\mathbf{x}.$$

Ainsi, si l'on identifie les éléments de $\mathfrak{H}_{Q_\alpha}$ et $\mathfrak{H}_{Q_\alpha}$ qui se correspondent par Φ, V_{Q_α} s'identifie à V et l'opérateur A (de $\mathfrak{H}_{Q_\alpha}$ dans $\mathfrak{H}$), associé à Q_α, s'identifie à un opérateur A_α de $\mathfrak{H}$ tel que A_α permute à V et que $A_\alpha A_\alpha^* = Q_\alpha$.

Proposition 3. *Soient A et B des opérateurs dans l'espace $\mathfrak{H}$, permutant à une translation unilatérale V de $\mathfrak{H}$. Supposons de plus que A est isométrique, B est une contraction, et que*

$$BB^* \geqq AA^*. \tag{8}$$

Il existe alors une isométrie C dans $\mathfrak{H}$, permutant à V et telle que $A = BC$.

Démonstration. Grâce à (8) et puisque B est une contraction, on a $I - AA^* \geqq$ $\geqq I - BB^* \geqq O$, d'où il s'ensuit que $(I - AA^*)h = 0$ (pour un $h \in \mathfrak{H}$) entraîne $(I - BB^*)h = 0$. Puisque A est isométrique, on a en particulier $(I - AA^*)Ag =$ $= Ag - AA^*Ag = Ag - Ag = 0$ pour tout $g \in \mathfrak{H}$, donc on a aussi $(I - BB^*)Ag = 0$, $Ag = BB^*Ag$. Ainsi, en posant $C = B^*A$, on a $A = BC$. Puisque A est isométrique et B, C sont des contractions, C est nécessairement isométrique elle aussi.

Comme A et B permutent à V, on a

$$V^*CV = V^*B^*AV = (BV)^*(AV) = (VB)^*(VA) = B^*V^*VA = C$$

d'où

$$(VV^*)CV = VC. \tag{9}$$

Or, C et V étant isométriques, il en est de même de CV et VC. Comme, d'autre part, VV^* est une projection orthogonale (notamment sur $V\mathfrak{H}$), on conclut de (9) que $CV = VC$, ce qui achève la démonstration.

2. Soit A un opérateur isométrique, non unitaire, dans l'espace $\mathfrak{H}$. Supposons que A permute à une translation unilatérale V de $\mathfrak{H}$, de multiplicité *infinie*.

Posons $Q = AA^*$ et $Q_\alpha = \alpha Q + (1-\alpha)I$ où $0<\alpha<1$. D'après la proposition 2, il existe alors un opérateur B_α dans $\mathfrak{H}$, permutant à V et tel que $B_\alpha B_\alpha^* = Q_\alpha$; puisque $O \leqq Q_\alpha \leqq I$, B_α est une contraction de $\mathfrak{H}$. Comme $B_\alpha = \alpha Q + (1-\alpha)I \geqq Q$, on a $B_\alpha B_\alpha^* \geqq AA^*$. En vertu de la proposition 3 il existe donc une isométrie C_α dans $\mathfrak{H}$ telle que C_α et V permutent que $A = B_\alpha C_\alpha$.

Montrons que ni B_α ni C_α n'est pas unitaire. En effet, l'équation $B_\alpha B_\alpha^* = I$ est impossible puisqu'elle entraîne $Q_\alpha = I$, donc $Q = I$, $AA^* = I$ et que par conséquent A (qui était supposé isométrique) est aussi unitaire, ce qui contredit l'hypothèse. D'autre part, si C_α était unitaire, $B_\alpha = A_\alpha C_\alpha^*$ vérifierait la relation $B_\alpha B_\alpha^* = AC_\alpha^* C_\alpha A^* = AA^*$, donc on aurait $Q_\alpha = Q$ et par conséquent de nouveau $Q = I$, ce qui est impossible.

Montrons, finalement, que $B_\alpha \mathfrak{H}$ est dense dans $\mathfrak{H}$. En cas contraire il y aurait un $h \neq 0$ tel que $B_\alpha^* h = 0$, $Q_\alpha h = B_\alpha B_\alpha^* h = 0$. Comme $Q_\alpha \geqq (1-\alpha)I$ on conclut que $(1-\alpha)h = 0$, $h = 0$. Contradiction.

Résumons le résultat que nous venons d'obtenir:

Proposition 4. *Tout opérateur isométrique non unitaire A d'un espace de Hilbert $\mathfrak{H}$, qui permute à une translation unilatérale V de $\mathfrak{H}$ de multiplicité infinie, peut être factorisée en produit $A = BC$ de deux opérateurs non unitaires de $\mathfrak{H}$ permutant à V, dont C est isométrique et B est une contraction telle que $\overline{B\mathfrak{H}} = \mathfrak{H}$.*

Il convient d'ajouter la suivante

Remarque. *Soit T un opérateur dans $\mathfrak{H}$, qui permute à la translation unilatérale V dans $\mathfrak{H}$. T applique $\mathfrak{H}$ unitairement sur $\mathfrak{H}$ si, et seulement si $T|\mathfrak{A}$ applique $\mathfrak{A}$ unitairement sur $\mathfrak{A}$, où $\mathfrak{A} = \mathfrak{H} \ominus V\mathfrak{H}$.*

En effet, si T est unitaire dans $\mathfrak{H}$, du fait que T permute à V il s'ensuit que T permute aussi à V^* et par conséquent à $P_\mathfrak{A} = I - VV^*$, donc $\mathfrak{A}$ réduit T, $T|\mathfrak{A}$ étant alors nécessairement unitaire dans $\mathfrak{A}$. Inversement, si $T|\mathfrak{A}$ applique $\mathfrak{A}$ unitairement sur $\mathfrak{A}$, $T|V^n\mathfrak{A}$ applique $V^n\mathfrak{A}$ unitairement sur $V^n\mathfrak{A}$ (puisque T et V permutent) et par conséquent T applique $\mathfrak{H} = \bigoplus_0^\infty V^n\mathfrak{A}$ unitairement sur $\mathfrak{H}$.

3. Soit $\mathfrak{E}$ un espace de Hilbert séparable et soit $H^2(\mathfrak{E})$ l'espace de Hilbert des fonctions $u(\lambda) = \sum_0^\infty a_n \lambda^n$ $(a_n \in \mathfrak{E}, \sum_0^\infty \|a_n\|^2 < \infty)$ avec la norme

$$\|u\| = \left[\sum_0^\infty \|a_n\|^2\right]^{1/2}.$$

Nous plongeons $\mathfrak{E}$ dans $H^2(\mathfrak{E})$ en identifiant l'élément $e \in \mathfrak{E}$ à la fonction constante $u(\lambda) \equiv e$. L'opérateur V défini par

$$(Vu)(\lambda) = \lambda \cdot u(\lambda)$$

est une translation unilatérale dans $H^2(\mathfrak{E})$; en effet on a $H^2(\mathfrak{E}) = \bigoplus_{0}^{\infty} V^n\mathfrak{E}$. Donc V a sa multiplicité égale à $\dim \mathfrak{E}$.

Une contraction Θ dans $H^2(\mathfrak{E})$ permute à V s'il existe une fonction analytique contractive $\{\mathfrak{E}, \mathfrak{E}, \Theta(\lambda)\}$ telle que

$$(\Theta u)(\lambda) = \Theta(\lambda) u(\lambda) \qquad (|\lambda| < 1)$$

et dans ce cas seulement.

Par définition, $\Theta(\lambda)$ est une fonction *intérieure* si Θ est un opérateur isométrique dans $H^2(\mathfrak{E})$ et $\Theta(\lambda)$ est une fonction *extérieure* si $\Theta H^2(\mathfrak{E})$ est dense dans $H^2(\mathfrak{E})$. La fonction $\Theta(\lambda)$ est *constante unitaire* (c'est-à-dire égale, pour tout λ, à un même opérateur unitaire de $\mathfrak{E}$), si, et seulement si l'opérateur Θ de $H^2(\mathfrak{E})$ est unitaire. Tout cela s'ensuit sans peine de [2], n° 2, en plongeant $H^2(\mathfrak{E})$ de manière évidente dans $L^2(\mathfrak{E})$.[2])

Cela étant, supposons que $\dim \mathfrak{E} = \aleph_0$ et que $\{\mathfrak{E}, \mathfrak{E}, \Theta(\lambda)\}$ est *une fonction intérieure qui n'est pas une constante unitaire*. Comme Θ est isométrique, non unitaire et permute à V, il s'ensuit de la proposition 4 qu'il existe une factorisation $\Theta = \Theta_2\Theta_1$ en produit de deux opérateurs de $H^2(\mathfrak{E})$, permutant à V, non unitaires, et tels que Θ_1 est isométrique et Θ_2 est une contraction ayant la propriété que $\Theta_2 H^2(\mathfrak{E})$ est dense dans $H^2(\mathfrak{E})$. Ces opérateurs sont alors engendrés par des fonctions analytiques contractives $\{\mathfrak{E}, \mathfrak{E}, \Theta_i(\lambda)\}$ $(i = 1, 2)$ dont $\Theta_1(\lambda)$ est une fonction *intérieure*, $\Theta_2(\lambda)$ est une fonction *extérieure*, aucune d'elles n'étant une constante unitaire.

Ainsi, nous avons obtenu le suivant

Théorème 1. *Lorsque $\mathfrak{E}$ est un espace de Hilbert de dimension $\aleph_0$, toute fonction analytique intérieure $\{\mathfrak{E}, \mathfrak{E}, \Theta(\lambda)\}$ qui n'est pas constante unitaire, peut être factorisée sous la forme*

$$\Theta(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda) \qquad (|\lambda| < 1) \tag{10}$$

en produit de deux fonctions analytiques contractives $\{\mathfrak{E}, \mathfrak{E}, \Theta_i(\lambda)\}$ $(i = 1, 2)$, dont $\Theta_1(\lambda)$ est intérieure, $\Theta_2(\lambda)$ est extérieure, et aucune n'est constante unitaire.

Remarques. Dans le cas où $\dim \mathfrak{E}$ est *finie*, telle factorisation *n'est pas possible.* En effet, $\Theta(\lambda)$ et $\Theta_1(\lambda)$ étant des fonctions intérieures, leurs limites radiales $\Theta(e^{it})$, $\Theta_1(e^{it})$ sont des isométries dans $\mathfrak{E}$, presque partout; vu que $\mathfrak{E}$ est de dimension finie, ces isométries sont nécessairement des opérateurs unitaires dans $\mathfrak{E}$. Il s'ensuit que $\Theta_2(e^{it}) = \Theta(e^{it})\Theta_1(e^{it})^{-1}$ est aussi un opérateur unitaire dans $\mathfrak{E}$, presque partout, d'où il résulte que Θ_2 est une isométrie dans $H^2(\mathfrak{E})$. Comme $\Theta_2(\lambda)$ est une fonction extérieure, $\Theta_2 H^2(\mathfrak{E})$ est dense dans $H^2(\mathfrak{E})$. Ainsi, l'opérateur isométrique Θ_2 est même unitaire. Cela est impossible puisqu'on a supposé que $\Theta_2(\lambda)$ n'est pas constante unitaire.

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[2]) On étend l'opérateur Θ à $L^2(\mathfrak{E})$ moyennant la définition

$$\tilde{\Theta}\Big(\sum_{|n| \le N} a_n e^{int}\Big) = e^{-iNt}\,\Theta\Big(\sum_{|n| \le N} a_n e^{i(n+N)t}\Big)$$

et on tient compte (pour la dernière assertion concernant Θ) de la remarque à la fin du n° précédent.

Ainsi, notre théorème marque des différences essentielles qu'on rencontre, dans le problème des factorisations des fonctions intérieures $\{\mathfrak{E}, \mathfrak{E}, \Theta(\lambda)\}$, lorsqu'on passe du cas où dim $\mathfrak{E}$ est finie au cas où dim $\mathfrak{E}$ est infinie.

Remarquons aussi que si la fonction $\Theta(\lambda)$ figurant dans le théorème est la fonction caractéristique d'une contraction T (aux indices de défaut égaux à $\aleph_0$ et telle que $T^{*n} \to O$; *cf.* [1]), la factorisation (10) ne correspond pas à un sous-espace invariant pour T parce que, dans les factorisations des fonctions caractéristiques intérieures qui correspondent à des sous-espaces invariants, les facteurs doivent aussi être intérieures (*cf.* [2], proposition 4.5).

4. Nous sommes à même de prouver notre

Théorème 2. *Toute fonction analytique contractive* $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ ($\mathfrak{E}, \mathfrak{E}_*$ *étant deux espaces de Hilbert séparables*), *qui n'est pas une constante unitaire, peut être factorisée en produit de deux fonctions analytiques contractives* $\{\mathfrak{E}, \mathfrak{F}, \Theta_1(\lambda)\}$ *et* $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_2(\lambda)\}$ *(où* $\mathfrak{F}$ *est aussi séparable), dont aucune n'est constante unitaire.*

Démonstration. En vertu de [2], § 5, n° 1, $\Theta(\lambda)$ admet les factorisations canoniques $\Theta(\lambda) = \Theta_i(\lambda)\Theta_e(\lambda) = \Theta_{*e}(\lambda)\Theta_{*i}(\lambda)$, où $\Theta_i(\lambda)$ est intérieure, $\Theta_e(\lambda)$ est extérieure, $\Theta_{*i}(\lambda)$ est $*$-intérieure, et $\Theta_{*e}(\lambda)$ est $*$-extérieure. Ainsi, ou bien $\Theta(\lambda)$ admet un facteur intérieur ou $*$-intérieur non-constant, ou bien elle est à la fois extérieure et $*$-extérieure. Dans le second cas, $\Theta(\lambda)$ admet une factorisation non banale en vertu de la proposition 5.3 de [2]. Il suffit donc d'envisager le cas où $\Theta(\lambda)$ est intérieure ou $*$-intérieure. Vu que pour une fonction $*$-intérieure $\Theta(\lambda)$, toute factorisation non banale $\Theta^{\tilde{}}(\lambda) = \Theta'^{\tilde{}}(\lambda)\Theta''^{\tilde{}}(\lambda)$ de la fonction intérieure $\Theta^{\tilde{}}(\lambda) = \Theta(\bar{\lambda})^*$ fournit la factorisation non banale $\Theta(\lambda) = \Theta''(\lambda)\Theta'(\lambda)$ pour $\Theta(\lambda)$, il nous reste à prouver le théorème dans le cas où $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ est une fonction intérieure.

Dans ce cas, dim $\mathfrak{E} \leqq$ dim $\mathfrak{E}_*$. Ainsi, on peut identifier $\mathfrak{E}$ à un sous-espace de $\mathfrak{E}_*$ et puis on peut plonger $\mathfrak{E}_*$ dans un espace $\mathfrak{F}$ de sorte que $\mathfrak{F} \ominus \mathfrak{E}_*$ soit aussi de dimension $\aleph_0$. Comme $\mathfrak{F} \ominus \mathfrak{E}$ et $\mathfrak{F} \ominus \mathfrak{E}_*$ ont alors la même dimension $\aleph_0$ il existe un opérateur partiellement isométrique Z dans $\mathfrak{F}$ qui applique $\mathfrak{F} \ominus \mathfrak{E}$ isométriquement sur $\mathfrak{F} \ominus \mathfrak{E}_*$ et tel que $Z|\mathfrak{E} = O$. Posons

$$\hat{\Theta}(\lambda) = \Theta(\lambda)P_{\mathfrak{E}} + Z \tag{11}$$

où $P_{\mathfrak{E}}$ est la projection orthogonale de $\mathfrak{F}$ dans $\mathfrak{E}$, $P_{\mathfrak{E}} = I - Z^*Z$.

Nous avons alors pour tout $f \in \mathfrak{F}$

$$\|\hat{\Theta}(e^{it})f\|^2 = \|\Theta(e^{it})P_{\mathfrak{E}}f + Zf\|^2 = \|\Theta(e^{it})P_{\mathfrak{E}}f\|^2 + \|Zf\|^2 =$$
$$= \|P_{\mathfrak{E}}f\|^2 + \|Zf\|^2 = \big((I - Z^*Z)f, f\big) + (Z^*Zf, f) = \|f\|^2$$

en tout point t où $\Theta(e^{it})$ est isométrique, donc presque partout. Il en résulte que $\{\mathfrak{F}, \mathfrak{F}, \hat{\Theta}(\lambda)\}$ est une fonction analytique intérieure. Nous pouvons donc appliquer à cette fonction le théorème 1. Ainsi, il existe une factorisation

$$\hat{\Theta}(\lambda) = \hat{\Theta}_2(\lambda)\hat{\Theta}_1(\lambda) \qquad (|\lambda| < 1), \tag{12}$$

en fonctions analytiques contractives $\{\mathfrak{F}, \mathfrak{F}, \hat{\Theta}_i(\lambda)\}$ $(i = 1, 2)$, dont aucune n'est constante unitaire.

Puisque $\Theta(\lambda)$ applique $\mathfrak{E}$ dans $\mathfrak{E}_*$, il s'ensuit de (11) que $\Theta(\lambda)=P_{\mathfrak{E}_*}\hat{\Theta}(\lambda)|\mathfrak{E}$, où $P_{\mathfrak{E}_*}$ désigne la projection orthogonale de $\mathfrak{F}$ dans $\mathfrak{E}_*$. Ainsi, (12) entraîne

$$\Theta(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda) \qquad (|\lambda|<1), \tag{13}$$

où

$$\Theta_1(\lambda)=\hat{\Theta}_1(\lambda)|\mathfrak{E}, \quad \Theta_2(\lambda)=P_{\mathfrak{E}_*}\hat{\Theta}_2(\lambda).$$

De cette sorte, on a obtenu une factorisation de $\{\mathfrak{E}, \mathfrak{E}_*, \Theta(\lambda)\}$ en produit des facteurs $\{\mathfrak{E}, \mathfrak{F}, \Theta_1(\lambda)\}$ et $\{\mathfrak{F}, \mathfrak{E}_*, \Theta_2(\lambda)\}$ qui sont évidemment des fonctions analytiques contractives. Montrons qu'aucune d'elles n'est une constante unitaire.

Observons d'abord, à cet effet, que pour $g\in\mathfrak{F}\ominus\mathfrak{E}$ on a

$$\|g\|\geqq\|\hat{\Theta}_1(\lambda)g\|\geqq\|\hat{\Theta}_2(\lambda)\hat{\Theta}_1(\lambda)g\|=\|\hat{\Theta}(\lambda)g\|=\|Zg\|=\|g\|$$

donc $\|\hat{\Theta}_1(\lambda)g\|=\|g\|$. Supposons que $\Theta_1(\lambda)\equiv\Theta_{10}$ où Θ_{10} est un opérateur unitaire de $\mathfrak{E}$ sur $\mathfrak{F}$. On a alors, pour $f\in\mathfrak{E}$, $g\in\mathfrak{F}\ominus\mathfrak{E}$,

$$\begin{aligned}\|f\|^2+\|g\|^2&=\|f+g\|^2\geqq\|\hat{\Theta}_1(\lambda)(f+g)\|^2=\\&=\|\Theta_{10}f+\hat{\Theta}_1(\lambda)g\|^2=\|\Theta_{10}f\|^2+\|\hat{\Theta}_1(\lambda)g\|^2+2\operatorname{Re}(\Theta_{10}f, \hat{\Theta}_1(\lambda)g)=\\&=\|f\|^2+\|g\|^2+2\operatorname{Re}(\Theta_{10}f, \hat{\Theta}_1(\lambda)g),\end{aligned}$$

donc $\operatorname{Re}(\Theta_{10}f, \hat{\Theta}_1(\lambda)g)\leqq 0$. Comme cela est vrai pour f aussi bien que pour εf (ε complexe, arbitraire), on a $(\Theta_{10}f, \hat{\Theta}_1(\lambda)g)=0$, et comme $\Theta_{10}\mathfrak{E}=\mathfrak{F}$, on a nécessairement $\hat{\Theta}_1(\lambda)g=0$, $\|g\|=\|\hat{\Theta}_1(\lambda)g\|=0$. Cela veut dire que $\mathfrak{F}\ominus\mathfrak{E}=\{0\}$, $\mathfrak{E}=\mathfrak{F}$ et par conséquent $\hat{\Theta}_1(\lambda)\equiv\Theta_1(\lambda)\equiv\Theta_{10}$, ce qui contredit ce que $\hat{\Theta}_1(\lambda)$ n'est pas constante unitaire. Donc le facteur $\Theta_1(\lambda)$ dans (13) n'est pas constant unitaire.

Passons au facteur $\Theta_2(\lambda)$ et supposons que $\Theta_2(\lambda)\equiv\Theta_{20}$ où Θ_{20} est un opérateur unitaire de $\mathfrak{F}$ sur $\mathfrak{E}_*$. Comme d'autre part $\Theta_2(\lambda)=P_{\mathfrak{E}_*}\hat{\Theta}_2(\lambda)$, où $\hat{\Theta}_2(\lambda)$ est une contraction, on a nécessairement $\hat{\Theta}_2(\lambda)\equiv\Theta_2(\lambda)\equiv\Theta_{20}$. Cela contredit ce que $\hat{\Theta}_2(\lambda)$ n'est pas constante unitaire. Donc $\Theta_2(\lambda)$ n'est pas constante unitaire.

Cela achève la démonstration du théorème 2.

Remarque. Dans une factorisation de type (13) il est essentiel d'admettre que l'espace intermédiaire $\mathfrak{F}$ puisse être choisi indépendamment de $\mathfrak{E}$ et de $\mathfrak{E}_*$. Par exemple, la fonction λ n'admet pas des factorisations non banales en produit de fonctions analytiques numériques, mais on a bien

$$\lambda=\Theta_2(\lambda)\Theta_1(\lambda) \quad \text{où} \quad \Theta_2(\lambda)=\left(\frac{1}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}}\right) \quad \text{et} \quad \Theta_1(\lambda)=\begin{pmatrix}\dfrac{\lambda}{\sqrt{2}}\\[2mm] \dfrac{1}{\sqrt{2}}\end{pmatrix}$$

(donc $\mathfrak{E}=\mathfrak{E}_*=E^1$ et $\mathfrak{F}=E^2$).

Ouvrages cités

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The Riemann—Lebesgue theorem on groups

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In [2] HEWITT gives an interesting and elegant *constructive* proof of PLANCHEREL's theorem for L^2 functions on a locally compact abelian group ($LCAG$). His proof is modelled on the classical proof of F. RIESZ of the special case in which the group is the real line R^1. We notice that in HEWITT's proof the Riemann—Lebesgue theorem is taken as known. However, to our knowledge, no *constructive* proof of the Riemann—Lebesgue theorem for the general $LCAG$ has ever been given. The proof of this theorem is easy in the case of R^1, but only because the explicit form of the group characters as functions is known. The theorem for the general $LCAG$ is always deduced from the Gelfand theory (see, for example, [4]) via the Tychonoff—Alaoglu theorem and other far from trivial considerations. This approach completely obscures the relation of the group structure to the theorem. In this paper we give a constructive proof of the Riemann—Lebesgue theorem for the general $LCAG$ (again modelled on a well-known proof of the case of R^1). In particular, some light will be thrown on the behavior of the group characters as functions. (See Definition B and Theorem H.)

We begin with the following well-known proof. (See [1], for example.)

Theorem A. (Riemann—Lebesgue theorem for R^1.) *Let $f \in L^1(R^1)$ and let $f^\wedge$ be the Fourier transform of f;*

$$f^\wedge(\gamma) = \int_{-\infty}^{\infty} e^{-i\gamma x} f(x)\, dx \qquad (\gamma \in R^1). \tag{1}$$

Then $\lim\limits_{\gamma \to \pm\infty} f^\wedge(\gamma) = 0$. *That is, the Fourier transform of an L^1 function vanishes at infinity.*

Proof. For $y \in R^1$ let $f_y(x) = f(x-y)$ $(y \in R^1)$. Given $\varepsilon > 0$ choose $\delta > 0$ such that $\|f - f_y\|_1 < 2\varepsilon$ if $|y| < \delta$. From (1) we have, for $\gamma \neq 0$,

$$-f^\wedge(\gamma) = \int_{-\infty}^{\infty} e^{-i\gamma\left(x+\frac{\pi}{\gamma}\right)} f(x)\, dx = \int_{-\infty}^{\infty} e^{-i\gamma x} f\left(x - \frac{\pi}{\gamma}\right) dx. \tag{2}$$

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Subtracting (2) from (1) we have

$$2f^\wedge(\gamma) = \int_{-\infty}^{\infty} e^{-i\gamma x}\left[f(x)-f\left(x-\frac{\pi}{\gamma}\right)\right] dx, \quad 2|f^\wedge(\gamma)| \leqq \|f-f_{\frac{\pi}{\gamma}}\|_1,$$

and hence, if $|\pi/\gamma|<\delta$, then $2|f^\wedge(\gamma)|<2\varepsilon$. That is,

$$|f^\wedge(\gamma)|<\varepsilon \qquad \left(|\gamma|>\frac{\pi}{\delta}\right).$$

This proves the theorem.

As we shall demonstrate, the key to the proof of the theorem is the fact that at the point $x=\frac{\pi}{\gamma}$, the character $x\to e^{i\gamma x}$ takes the value -1. In particular, if U is the neighborhood $(-\delta, \delta)$ of 0, then there is a compact set $K=\left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right]$ such that if $\gamma\in R^1-K$ then the character determined by γ (namely $x\to e^{i\gamma x}$) takes a value at some point of U (namely π/γ) whose real part is $\leqq 0$. It is this property that we shall demonstrate for the general *LCAG*.

Definition B. Let G be a *LCAG* with character group Γ. We say that G has the $\mathrm{R-L}$ *property*, if, for any neighborhood U of the identity 0 of G there exists a compact set K in Γ such that, if $\gamma\in\Gamma-K$ then there exists $x\in U$ with $\operatorname{Re}\gamma(x)\leqq 0$. (We call K a *compact set corresponding to* U.)

As we have seen, R^1 has the $\mathrm{R-L}$ property. It is now easy to show that if the *LCAG* G has the $\mathrm{R-L}$ property then the Riemann—Lebesgue theorem holds for G.

Theorem C. *Let G be a LCAG with the* $\mathrm{R-L}$ *property. If $f\in L^1(G)$ and $f^\wedge$ is the Fourier transform of f, i.e.*

$$f^\wedge(\gamma) = \int_G \overline{\gamma(x)}f(x)\,dx \qquad (\gamma\in\Gamma), \tag{3}$$

then $f^\wedge$ vanishes at infinity.

Proof. We simply imitate the proof in Theorem A. Given $\varepsilon>0$ choose a neighborhood U of 0 in G such that $\|f-f_y\|_1<\varepsilon$ if $y\in U$. (Here again, $f_y(x)=f(x-y)$.) According to the $\mathrm{R-L}$ property there exists a compact set K in Γ corresponding to U. Then if $\gamma\in\Gamma-K$ there exists x_0 in U with $\operatorname{Re}\gamma(x_0)\leqq 0$. So

$$\overline{\gamma(x_0)}f^\wedge(\gamma) = \int_G \overline{\gamma(x+x_0)}f(x)\,dx = \int \overline{\gamma(x)}f_{x_0}(x)\,dx, \tag{4}$$

and, subtracting (4) from (3),

$$|f^\wedge(\gamma)|\cdot|1-\overline{\gamma(x_0)}| \leqq \|f-f_{x_0}\|_1<\varepsilon.$$

Since $\operatorname{Re}\gamma(x_0)\leqq 0$ we must have $|1-\overline{\gamma(x_0)}|\geqq 1$. Thus $|f^\wedge(\gamma)|<\varepsilon$ for all $\gamma\in\Gamma$ outside of the compact set K. That is, $f^\wedge$ vanishes at infinity, which is what we wished to show.

In view of Theorem C, to show that the Riemann—Lebesgue theorem holds for an arbitrary *LCAG* G, it is sufficient to show that G has the $\mathrm{R-L}$ property. We do this in several steps ultimately making use of structure theory for the *LCAG*.

Lemma D. *If each of the locally compact abelian groups* G_1 *and* G_2 *has the* $\mathrm{R-L}$ *property, then so does* $G_1 \times G_2$.

Proof. Let $G = G_1 \times G_2$. Then the character group Γ of G is $\Gamma_1 \times \Gamma_2$ where Γ_i is the character group of G_i. Let U be any neighborhood of the identity in G. We may assume that $U = U_1 \times U_2$ where U_i is a neighborhood of the identity 0_i in G_i. According to the $\mathrm{R-L}$ property for G_i $(i=1, 2)$, there exists a compact subset K_i of Γ_i corresponding to U_i. Now let $K = K_1 \times K_2$. If $\gamma = (\gamma_1, \gamma_2) \in \Gamma - K$ then either $\gamma_1 \notin K_1$ or $\gamma_2 \notin K_2$. We may assume $\gamma_1 \notin K_1$. Then, by the $\mathrm{R-L}$ property for G_1, there exists $x_1 \in U_1$ with $\mathrm{Re}\ \gamma_1(x_1) \leqq 0$. Let $x = (x_1, 0_2)$. Then $x \in U$ and $\gamma(x) = \gamma_1(x_1)\gamma_2(0_2) = \gamma_1(x_1)$ and hence $\mathrm{Re}\ \gamma(x) \leqq 0$. Thus K may be used as a compact set corresponding to U, and so G has the $\mathrm{R-L}$ property.

Next we shall show that for any *compact* abelian group G, the topology for G is generated by finite *independent* subsets of Γ. (The subset $\{\beta_1, \dots, \beta_s\}$ of elements of a group is said to be independent if whenever $n_1, \dots, n_s$ are integers with $\sum_{i=1}^{s} n_i \beta_i = 0$ then $n_i \beta_i = 0$ for all $i = 1, \dots, s$.)

If $C = \{\gamma_1, \dots, \gamma_n\}$ is a finite set of characters of G and S is a symmetric open arc of the unit circle about 1 (that is, for some θ_0 with $0 < \theta_0 \leqq \pi$, $S = \{e^{i\theta} \mid -\theta_0 < \theta < \theta_0\}$) then $U[C; S]$ denotes the set of x in G such that $\gamma_k(x) \in S$ for $k = 1, \dots, n$. It is well known that the collection of all such $U[C: S]$ forms a basic set of neighborhoods of 0 in G. Thus, to show that the finite *independent* sets in Γ generate the topology of G, it is enough to show

Lemma E. *Let* G *be a compact abelian group. Then any neighborhood* $U[C; S]$ *contains a neighborhood* $U[B; S']$ *where* B *is a finite independent subset of* Γ.

Proof. Let $[C]$ denote the subgroup of Γ generated by $C = \{\gamma_1, \dots, \gamma_n\}$. Then $[C]$ is a finitely generated abelian group and is thus the direct sum of s cyclic subgroups. Let $\beta_1, \dots, \beta_s$ be the generators of these cyclic subgroups. Then $B = \{\beta_1, \dots \beta_s\}$ is an independent set. Now any γ_k in C may be expressed as $\gamma_k = n_{k1}\beta_1 + \dots + n_{ks}\beta_s$ where the n_{kj} are integers. (These representations may not be unique since some β's can have finite order. In any case for each $k = 1, 2, \dots, n$, fix one such representation.) Let $M_k = \sum_{j=1}^{s} |n_{kj}|$ and let $M = \max_{1 \leqq k \leqq n} M_k$. If $S = \{e^{i\theta} \mid -\theta_0 < \theta < \theta_0\}$ let $S' = \{e^{i\theta} \mid -\theta_0' < \theta < \theta_0'\}$ where $\theta_0' = \dfrac{\theta_0}{M}$. We shall show that $U[B; S'] \subset U[C; S]$. Indeed, if $x \in U[B; S']$ then $\beta_j(x) = e^{i\theta_j}$ where $|\theta_j| < \theta_0'$. Hence $\gamma_k(x) = \exp(i[n_{k1}\theta_1 + \dots + n_{ks}\theta_s])$. But $|n_{k1}\theta_1 + \dots + n_{ks}\theta_s| < M_k\theta_0' \leqq M\theta_0' = \theta_0$. Hence $\gamma_k(x) \in S$ for $k = 1, \dots, n$ and so $x \in U[C; S]$ which is what we wished to show.

We next prove

Lemma F. *Every compact abelian group* G *has the* $\mathrm{R-L}$ *property.*

Proof. Let U be any neighborhood of 0 in G. By the preceding lemma we may assume that $U=U[B; S]$ where $B=\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$ is an independent set, the α_j having finite order and the β_j infinite order, and $S=\{e^{i\theta} | -\theta_0<\theta<\theta_0\}$ where $0<\theta_0 \leqq \pi$. Let q be the smallest positive integer such that $q\theta_0 > \frac{\pi}{2}$. (Then $\operatorname{Re} e^{iq\theta_0} \leqq 0$.)

Let K be the (finite) set of γ in Γ which can be expressed $\gamma=\alpha+n_1\beta_1+\ldots+n_s\beta_s$ where α is an element of $[\alpha_1, \ldots, \alpha_r]$, the finite subgroup generated by $\alpha_1, \ldots, \alpha_r$, and $|n_1|+\ldots+|n_s| \leqq q$. (If there are no α_j — that is if every element in B has infinite order — use $\{0\}$ instead of $[\alpha_1, \ldots, \alpha_r]$. If there are no β_j, set $K=[\alpha_1, \ldots, \alpha_r]$ and use obvious modifications in the remainder of the proof.) We shall now show that K may be taken as a compact set corresponding to U. For suppose $\gamma \in \Gamma - K$. There are two possible cases.

I. Suppose $\gamma \in [B]$ where $[B]$ is the subgroup of Γ generated by B. Then $\gamma=\alpha+ +n_1\beta_1+\ldots+\ldots+n_s\beta_s$ for some $\alpha \in [\alpha_1, \ldots, \alpha_r]$ and for some integers $n_1, \ldots, n_s$. Since $\gamma \notin K$ we must have $M=|n_1|+\ldots+|n_s|>q$. For $j=1, \ldots, r$ let $x(\alpha_j)=1$. For $j=1, \ldots, s$ let $x(\beta_j)=e^{iq\theta_0/M}$ if $n_j>0$, let $x(\beta_j)=e^{-iq\theta_0/M}$ if $n_j<0$, and let $x(\beta_j)=1$ if $n_j=0$. Since $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are independent it is easy to verify that x may be extended multiplicatively to a character on $[B]$. We then have $x(\gamma)= =\exp\left((iq\theta_0/M)[|n_1|+\ldots+|n_s|]\right)=\exp(iq\theta_0)$. Now $[B]$ is a closed subgroup of Γ (since Γ is discrete). Hence we may extended x to a character on all of Γ, that is $x \in G$. But since $q\theta_0/M<\theta_0$ we have $\beta_j(x)=x(\beta_j) \in S$ for $j=1, \ldots, s$. Thus $x \in U$. Since $\operatorname{Re} \gamma(x)=\operatorname{Re} x(\gamma)=\operatorname{Re} e^{iq\theta_0} \leqq 0$, this shows that γ takes an appropriate value at x. (Note: if there are no β_j in B then case I cannot occur, since then $K=[B]$ and $\gamma \notin K$.)

II. Suppose $\gamma \notin [B]$. Then there is an element $y \in G$ such that y is in the annihilator of $[B]$ but $y(\gamma)=\gamma(y) \neq 1$. If $x=y^p$ for an appropriate positive integer p, we have $\operatorname{Re} \gamma(x)=\operatorname{Re}[\gamma(y)]^p \leqq 0$. But x is also in the annihilator of $[B]$ so that $\alpha_j(x)=x(\alpha_j)= =1=x(\beta_j)=\beta_j(x)$ for all $\alpha_j, \beta_j \in B$. Hence, $x \in U$ and the proof is complete.

Lemma G. *Let H be a LCAG which contains a compact open subgroup G. Then H has the* R$-$L *property.*

Proof. Let U be any neighborhood of the identity 0 of H. We may assume $U \subset G$. (Otherwise, since G is open, we could consider $U \cap G$ instead of U.) Lemma F shows that the compact group G has the R$-$L property. Thus there is a finite subset $K_0=\{\gamma_1, \ldots, \gamma_n\}$ of characters of G corresponding to U. Since G is compact, every γ_j may be extended to a character λ_j of H. If μ_j is any other character of H which is also an extension of γ_j then $\mu_j\lambda_j^{-1}$ is an element of the annihilator A of G.

(Here, of course, $A \subset H^\wedge$ where $H^\wedge$ is the character group of H.) That is, $\mu_j \in \lambda_j A$. Hence, if we set $K=\bigcup_{j=1}^{n} \lambda_j A$ then K is the set of all extensions of $\gamma_1, \ldots, \gamma_n$ to characters of H. Moreover, K is compact since A, being the annihilator of the open compact group G, is itself open and compact [3]. It is now easy to show that K may be used as a compact set corresponding to U. For if $\lambda \in H^\wedge - K$, then λ_G (the restriction of λ to G) is not one of the γ_j. That is, $\lambda_G \in G^\wedge - K_0$. Thus there exists $x \in U$ with $\operatorname{Re} \lambda_G(x) \leqq 0$. Obviously, then, $\operatorname{Re} \lambda(x) \leqq 0$ and we are done.

We now conclude with

Theorem H. *Every LCAG has the* R $-$ L *property.*

Proof. Every *LCAG* may be factored as $R^n \times H$ for some $n=0, 1, 2, \ldots$, where R^n is Euclidean n-space and H is a *LCAG* with a compact open subgroup [3]. After Definition B we observed that R^1 has the R $-$ L property. Hence, by Lemma D, R^n also has the R $-$ L property. This together with Lemma G and another application of Lemma D complete the proof.

Corollary I. *The Riemann—Lebesgue theorem holds for every LCAG.*

Proof. Theorem H and Theorem C.

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Ein Gleichverteilungssatz für Systeme homogener Linearformen modulo p

Von L. RÉDEI in Szeged und H. J. WEINERT in Potsdam

Professor Ott-Heinrich Keller zum 60. Geburtstag gewidmet

1. Es sei p eine Primzahl und $R = \mathfrak{J}/(p)$ der Restklassenring des Ringes $\mathfrak{J}$ der ganzen Zahlen modulo p, für dessen Elemente wir der Kürze halber oft einfach $0, 1, \ldots, p-1$ schreiben. Ein System von r homogenen Linearformen in n Unbestimmten

(1) $$L_\varrho(x_1, x_2, \ldots, x_n) = \sum_{\nu=1}^{n} a_{\varrho,\nu} x_\nu \in R[x_1, x_2, \ldots, x_n] \qquad (\varrho = 1, \ldots, r)$$

nennen wir (modulo p) *gleichverteilt*, wenn $p|r$ gilt und für jedes n-tupel

(2) $$(\xi_1, \xi_2, \ldots, \xi_n) \neq (0, 0, \ldots, 0) \quad \text{aus} \quad R^{[n]} = R \times R \times \ldots \times R$$

die r Werte

(3) $$L_\varrho(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{\nu=1}^{n} a_{\varrho,\nu} \xi_\nu \qquad (\varrho = 1, \ldots, r)$$

gleichverteilt sind, d. h. alle Restklassen modulo p gerade $\frac{r}{p}$-mal ergeben. Weiterhin bezeichnen wir als *volles System* in n Unbestimmten das aus allen $r = p^n$ paarweise voneinander verschiedenen Linearformen bestehende System

(4) $$L_{i_1 i_2 \ldots i_n}(x_1, x_2, \ldots, x_n) = i_1 x_1 + i_2 x_2 + \ldots + i_n x_n,$$

wobei also $(i_1, i_2, \ldots, i_n)$ die Produktmenge $R^{[n]}$ durchläuft.

Es ist klar, daß das volle System (4) gleichverteilt ist. Ausgangspunkt der vorliegenden Note war die Frage, ob umgekehrt ein gleichverteiltes System (1) von $r = p^n$ Linearformen das volle System sein muß. In der Tat gilt sogar der folgende, etwas schärfere

Satz. *Ein System* (1) *von r homogenen Linearformen in n Unbestimmten modulo p ist genau dann gleichverteilt, wenn $p^n | r$ gilt und es aus $s = \frac{r}{p^n}$ vollen Systemen besteht.*

Darüber hinaus zeigen wir, daß dieser Satz weder durch eine (wesentliche) Abschwächung der Gleichverteilung noch durch eine Ausdehnung auf andere Polynome aus $R[x_1, \ldots, x_n]$ verallgemeinert werden kann. Schließlich bemerken wir, daß sich unser Satz offenbar auch in die folgende Aussage überführen läßt:

Zwei Systeme (1) von je r Linearformen modulo p liefern genau dann für jedes n-tupel (2) die gleichen Werte, wenn diese Systeme übereinstimmen.

2. Beweis des Satzes. Wir brauchen nur zu zeigen, daß jedes gleichverteilte System aus einem oder mehreren vollen Systemen besteht. Für $n=1$ ist diese Behauptung trivial; wir zeigen, daß sie für ein System (1) in $n \geqq 2$ Unbestimmten richtig ist, falls sie für Systeme in $n-1$ Unbestimmten zutrifft. Die Einsetzung von $(1, 0, \ldots, 0)$ in (1) lehrt zunähst, daß die Koeffizienten $a_{\varrho,1}$ gleichverteilt sind; ohne Beschränkung der Allgemeinheit können wir daher das System (1) in der Form

$$L_\tau^{(i)}(x_1, x_2, \ldots, x_n) = ix_1 + \sum_{\nu=2}^{n} a_{\tau,\nu}^{(i)} x_\nu \tag{5}$$

schreiben, wobei i die Klassen $0, 1, \ldots, p-1$ und τ jeweils alle Werte $1, \ldots, t$ mit $r=pt$ durchläuft. Wir betrachten nun für jedes feste $(n-1)$-tupel $(\xi_2, \ldots, \xi_n) \neq (0, \ldots, 0)$ die $p^2 t$ Werte

$$L_\tau^{(i)}(\xi_1, \xi_2, \ldots, \xi_n) = i\xi_1 + \sum_{\nu=2}^{n} a_{\tau,\nu}^{(i)} \xi_\nu \tag{6}$$

für alle i, alle τ und alle $\xi_1 \in R$, die nach Voraussetzung für jeden Wert von ξ_1, also auch insgesamt gleichverteilt sind. Ist $c \in R$ fest gewählt, so trifft die gleiche Feststellung für die $p^2 t$ Werte

$$L_\tau^{(i)}(\xi_1, \xi_2, \ldots, \xi_n) - c\xi_1 = (i-c)\xi_1 + \sum_{\nu=2}^{n} a_{\tau,\nu}^{(i)} = F(i, \tau, \xi_1) \tag{6'}$$

zu. Nun sind aber für jedes τ und jedes $i \neq c$ die p Werte $F(i, \tau, \xi_1)$ mit $\xi_1 = 0, 1, \ldots, \ldots, p-1$ gleichverteilt. Wir streichen sie alle aus (6'), womit auch die aus pt Werten bestehende Restmenge

$$F(c, \tau, \xi_1) = 0\xi_1 + \sum_{\nu=2}^{n} a_{\tau,\nu}^{(c)} \xi_\nu \tag{6''}$$

gleichverteilt ist. Da aber diese Werte von ξ_1 gar nicht mehr abhängen, so folgt, daß für jedes $c \in R$ die t Linearformen in $n-1$ Unbestimmten

$$\sum_{\nu=2}^{n} a_{\tau,\nu}^{(c)} x_\nu \qquad (\tau = 1, \ldots, t)$$

gleichverteilt sind. Nach Induktionsvoraussetzung bestehen sie also jeweils aus $s = \frac{t}{p^{n-1}}$ vollen Systemen in $n-1$ Unbestimmten und damit (5) aus $s = \frac{pt}{p^n} = \frac{r}{p^n}$ vollen Systemen in n Unbestimmten, wie zu beweisen war.

3. Für die weiteren Bemerkungen stellen wir zunächst fest, daß ersichtlich nicht alle $p^n - 1$ n-tupel (2) eingesetzt werden müssen, um nachzuprüfen, ob ein System (1) gleichverteilt ist. Vielmehr kann man sich (auf die verschiedenen Punkte des $n-1$-dimensionalen projektiven Raumes über R d.h.) etwa auf die folgenden

$1+p+\ldots+p^{n-1}=\dfrac{p^n-1}{p-1}$ n-tupel beschränken:

$$
(7)\qquad
\begin{array}{ll}
\left.\begin{array}{l}
(1\ ,0,0,\ldots\ldots\ldots,0,\ 0)\\
(\xi_1,1,0,\ldots\ldots\ldots,0,\ 0)\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} & \text{a)}\\
(\xi_1,\ldots,\xi_k,1,0,\ldots,0) & \text{b)}\\
\left.\begin{array}{l}
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(\xi_1,\xi_2,\xi_3,\ldots,1\quad,0)\\
(\xi_1,\xi_2,\xi_3,\ldots,\xi_{n-1},1)
\end{array}\right\} & \text{c)}
\end{array}
$$

Dagegen gilt unser Satz nicht mehr, wenn von diesen (oder auf andere Weise normierten) wesentlichen Einsetzungen eine weggelassen wird:

Zusatz. *Für jedes feste n-tupel* $\eta=(\eta_1,\ldots,\eta_k,1,0,\ldots,0)$ *von* (7) *gibt es stets ein System* (1) *von* $r=p^{n-1}$ *Linearformen in n Unbestimmten, welches beim Einsetzen von jedem n-tupel* (7) *mit Ausnahme von* η *gleichverteilte Werte liefert, nämlich etwa*

$$
(8)\qquad i_1x_1+\ldots+i_kx_k-(i_1\eta_1+\ldots+i_k\eta_k)x_{k+1}+i_{k+2}x_{k+2}+\ldots+i_nx_n
$$

mit allen $(i_1,\ldots,i_k,i_{k+2},\ldots,i_n)\in R^{[n-1]}$.

Beweis. Beim Einsetzen der in (7) unter a) bzw. c) zusammengefaßten n-tupel in (8) erhält man ersichtlich immer gleichverteilte Werte. Im Falle b) ergeben sich aus (8) mehrfach die Werte

$$
i_1(\xi_1-\eta_1)+\ldots i_k(\xi_k-\eta_k),
$$

die genau dann glechverteilt sind, wenn wenigstens für eine Differenz $\xi_\nu-\eta_\nu\neq 0$ gilt.

Aus diesem Zusammenhang folgt sofort, daß kein entsprechender Gleichverteilungssatz für inhomogene Linearformen oder Systeme von Polynomen höheren Grades gilt[1]). Jedem solchen System kann nämlich nach der Anzahl n der auftretenden Koeffizienten ein System (1) von homogenen Linearformen in n Unbestimmten zugeordnet werden, wobei jedoch wenigstens für eine Unbestimmte nicht mehr alle p Werte aus R eingesetzt werden können oder die Einsetzungsmöglichkeiten für verschiedene Unbestimmte nicht mehr unabhängig voneinander sind. Eine einzige (wesentliche) Ausnahme beim Einsetzen macht aber nach unserem Zusatz die Aussage des Gleichverteilungssatzes bereits falsch.

(Eingegangen am 3. Februar 1965)

[1]) Von Umschreibungen wie x^p für x usw., die beim Einsetzen modulo p trivialer Weise den gleichen Wert liefern, hat man dabei natürlich abzusehen.

Verallgemeinerung eines Satzes über homogene Linearformen

Von L. RÉDEI in Szeged und H. J. WEINERT in Potsdam

Die Begriffsbildungen der vorangehenden Arbeit [1] verallgemeinern wir auf folgende Weise: Es sei $m > 1$ eine ganze Zahl und $R = \mathfrak{J}/(m)$ der Restklassenring des Ringes $\mathfrak{J}$ der ganzen Zahlen modulo m. Wir betrachten Systeme von r homogenen Linearformen in n Unbestimmten

$$\left.\begin{aligned} &(1) \qquad L_\varrho(x_1, x_2, \ldots, x_n) = \sum_{\nu=1}^{n} a_{\varrho,\nu} x_\nu \\ &(2) \qquad K_\varrho(x_1, x_2, \ldots, x_n) = \sum_{\nu=1}^{n} b_{\varrho,\nu} x_\nu \end{aligned}\right\} \quad (\varrho = 1, \ldots, r)$$

aus $R[x_1, x_2, \ldots, x_n]$. Ein n-tupel

$$(3) \qquad (\xi_1, \xi_2, \ldots, \xi_n) \in R^{[n]} = R \times R \times \ldots \times R$$

heiße *zulässig*, wenn wenigstens ein ξ_ν kein Nullteiler von R ist. Wir nennen ein System (1) (modulo m) *gleichverteilt*, wenn $m|r$ gilt und für jedes zulässige n-tupel (3) die r Werte

$$(4) \qquad L_\varrho(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{\nu=1}^{n} a_{\varrho,\nu} \xi_\nu \qquad (\varrho = 1, \ldots, r)$$

gleichverteilt sind, d. h. alle Restklassen modulo m gerade $\frac{r}{m}$-mal ergeben. Dagegen sagen wir, daß die Systeme (1) und (2) *wertgleich* sind, wenn für jedes zulässige n-tupel (3) die r Werte $L_\varrho(\xi_1, \xi_2, \ldots, \xi_n)$ in ihrer Gesamtheit mit den r Werten $K_\varrho(\xi_1, \xi_2, \ldots, \xi_n)$ übereinstimmen. Weiterhin bezeichnen wir als *volles System* in n Unbestimmten das aus allen $r = m^n$ paarweise voneinander verschiedenen Linearformen von der Form (1) bestehende System.

Schließlich sagen wir, daß modulo m der *Gleichverteilungssatz* gilt, wenn *ein System* (1) *von r homogenen Linearformen in n Unbestimmten modulo m genau dann gleichverteilt ist, wenn $m^n|r$ gilt und es aus $s = \frac{r}{m^n}$ vollen Systemen besteht.* Dagegen sei modulo m der *Wertgleichheitssatz* erfüllt, wenn *zwei Systeme* (1) *und* (2) *von je r homogenen Linearformen in n Unbestimmten modulo m genau dann wertgleich sind, wenn diese Systeme übereinstimmen, also $L_\varrho(x_1, x_2, \ldots, x_n) = K_\varrho(x_1, x_2, \ldots, x_n)$ bei geeigneter Numerierung gilt.*

Für den Fall, daß der Modul $m=p$ eine Primzahl ist, haben wir in [1] den Gleichverteilungssatz bewiesen und den ihm entsprechenden Wertgleichheitssatz erwähnt. In dieser Note werden wir folgendes zeigen:

Satz 1. *Für jeden Modul m folgt aus dem Gleichverteilungssatz der Wertgleichheitssatz und umgekehrt.*

Satz 2. *Der Wertgleichheitssatz gilt für jeden Modul $m=p^\mu$, wobei p eine Primzahl ist.*

Bezüglich der Frage, ob Satz 2 auch für $m \neq p^\mu$ gilt, haben wir bisher nur Vermutungen.

Beweis von Satz 1. Als erstes sei modulo m der Gleichverteilungssatz erfüllt. Es genügt, die Annahme zum Widerspruch zu führen, daß zwei nichtidentische Systeme (1) und (2) modulo m wertgleich sind. Dabei möge in (1) ein und dieselbe Linearform maximal s-mal auftreten. Dann ist (1) ein Teilsystem von s vollen Systemen. Wechselt man in letzterem (1) gegen (2) aus, so entsteht ersichtlich wieder ein gleichverteiltes System von sm^n Linearformen, welches nicht aus s vollen Systemen besteht, im Widerspruch zum Gleichverteilungssatz.

Umgekehrt sei der Wertgleichheitssatz modulo m erfüllt. Wir nehmen jetzt an, daß es ein gleichverteiltes System (1) von r Linearformen modulo m gibt, welches nicht aus vollen Systemen besteht. Dieses System (1) nehmen wir in m^n Exemplaren und erhalten ein ebenfalls gleichverteiltes System von rm^n Linearformen, welches dann auch nicht aus vollen Systemen besteht. Es ist also von dem aus r vollen Systemen bestehenden System verschieden, aber mit diesem wertgleich. Dies widerspricht der Gültigkeit des Wertgleichheitssatzes.

Beweis von Satz 2. Wir schicken für $m=p^\mu$ folgende Hilfsüberlegungen voraus:

1. Die Anzahl der zulässigen n-tupel $(\xi_1, \xi_2, \dots, \xi_n)$ beträgt

$$P(n, \mu) = m^n - (m-\varphi(m))^n = (p^\mu)^n - (p^{\mu-1})^n = p^{n(\mu-1)}(p^n-1).$$

2. Es sei $a_1x_1 + a_2x_2 + \dots + a_nx_n \in R[x_1, x_2, \dots, x_n]$ eine Linearform, in der wenigstens ein Koeffizient nicht Nullteiler von R ist; ohne Beschränkung der Allgemeinheit gelte etwa $p \nmid a_1$. Dann gibt es genau

$$P(n-1, \mu) = p^{(\mu-1)(n-1)}(p^{n-1}-1)$$

zulässige n-tupel $(\xi_1, \xi_2, \dots, \xi_n)$, die

$$a_1\xi_1 + a_2\xi_2 + \dots + a_n\xi_n \equiv cp^\varkappa \pmod{p^\mu} \tag{5}$$

erfüllen, wobei $1 \leqq \varkappa \leqq \mu$ und $c \in \mathfrak{I}$ gilt. In der Tat gibt es zu jedem der $P(n-1, \mu)$ zulässigen $(n-1)$-tupeln $(\xi_2, \dots, \xi_n)$ genau ein ξ_1, welches (5) erfüllt. Dagegen führt ein nicht zulässiges $(n-1)$-tupel $(\xi_2, \dots, \xi_n)$ auch zu einem ξ_1 mit $p|\xi_1$, sodaß keine weiteren zulässigen Lösungen von (5) entstehen.

3. Es sei $a_1x_1 + a_2x_2 + \dots + a_nx_n \in R[x_1, x_2, \dots, x_n]$ eine Linearform mit $p^\tau | a_\nu$ $(1 \leqq \tau < \mu)$ für alle a_ν, aber $p^{\tau+1} \nmid a_\nu$ für wenigstens ein a_ν. Wir schreiben dafür auch $p^\tau \| (a_1, \dots, a_n)$, also p^τ ist der genaue p-Teiler des größten gemeinsamen Teilers

der a_ν. Dann gibt es genau $p^\tau P(n-1, \mu)$ zulässige n-tupel $(\xi_1, \xi_2, \ldots, \xi_n)$, die

(6) $$a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n \equiv 0 \pmod{p^\mu}$$

erfüllen. In der Tat erhalten wir mit $a_\nu = p^\tau a'_\nu$ für jedes c mit $0 \leqq c < p^\tau$ gemäß 2. und

(7) $$a'_1\xi_1 + a'_2\xi_2 + \ldots + a'_n\xi_n \equiv p^{\mu-\tau}c \bmod p^\mu$$

jeweils $P(n-1, \mu)$ n-tupel, die (6) erfüllen. Diese $p^\tau P(n-1, \mu)$ zulässigen n-tupel sind auch paarweise verschieden und die sämtlichen, welche (6) erfüllen. Letzteres erkennt man daraus, daß jede Lösung von (6) wegen

$$a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n = p^\tau p^{\mu-\tau}c$$

mit einem geeigneten $c \in \mathfrak{J}$ auch

$$a'_1\xi_1 + a'_2\xi_2 + \ldots + a'_n\xi_n = p^{\mu-\tau}c$$

erfüllt, also eine Lösung von (7) ist.

Der eigentliche Beweis unseres Satzes 2 beruht nun auf dem Gedanken, daß man von zwei wertgleichen, aber verschiedenen Systemen (1) und (2) zu ebensolchen übergehen könnte, bei denen die Linearform

$$0 = 0x_1 + 0x_2 + \ldots + 0x_n$$

zwar etwa in (1), aber nicht in (2) auftritt. Zunächst kann man nämlich erreichen, daß eine Linearform von (1), etwa L_1, nicht in (2) auftritt und dann zu den Systemen

$$L_\varrho - L_1 \quad \text{und} \quad K_\varrho - L_1 \qquad (\varrho = 1, \ldots, r)$$

übergehen. Weiterhin müssen wertgleiche Systeme, wenn in beiden alle zulässigen n-tupel $(\xi_1, \xi_2, \ldots, \xi_n)$ eingesetzt werden, gleich oft den Wert 0 ergeben.

Für $m = p$ [1]) kommen wir damit rasch zu einem Widerspruch, da die Linearformen

$$\sum a_\nu x_\nu \quad \text{von (1) bzw.} \quad \sum b_\nu x_\nu \quad \text{von (2)}$$

mit $p \nmid (a_1, \ldots, a_n)$ bzw. $p \nmid (b_1, \ldots, b_n)$ gemäß 2. nur für $P(n-1, 1) = p^{n-1} - 1$, die Linearform 0 aber für $p^n - 1$ zulässige n-tupel verschwinden. Für $m = p^\mu$ führen wir einen vollständigen Induktionsschluß nach μ durch, indem wir beide Systeme (1) und (2) modulo p^τ mit $\tau = 1, \ldots, \mu - 1$ betrachten. Die entstehenden Systeme sind auch modulo p^τ wertgleich, mithin nach Induktionsannahme identisch. Dies lehrt, daß in beiden Systemen jeweils gleichviele Linearformen auftreten, für die (mit von selbst klaren Bezeichnungen)

(8)
$$\begin{array}{ccc}
p \nmid (a_1, \ldots, a_n) & \text{bzw.} & p \nmid (b_1, \ldots, b_n) \\
p \| (a_1, \ldots, a_n) & \text{bzw.} & p \| (b_1, \ldots, b_n) \\
\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\
p^{\mu-2} \| (a_1, \ldots, a_n) & \text{bzw.} & p^{\mu-2} \| (b_1, \ldots, b_n) \\
p^{\mu-1} | (a_1, \ldots, a_n) & \text{bzw.} & p^{\mu-1} | (b_1, \ldots, b_n)
\end{array}$$

[1]) Natürlich ist der Fall $m = p$ schon in [1] erledigt worden.

erfüllt ist. In der letzten Zeile von (8) gilt dabei rechts sogar $p^{\mu-1}\|(b_1, \ldots, b_n)$, während links die Linearform 0 mit $p^{\mu}|(a_1, \ldots, a_n)$ wenigstens einmal auftritt. Die Anwendung von 3. ergibt, daß damit das System (1) für mehr n-tupel $(\xi_1, \xi_2, \ldots, \xi_n)$ verschwindet, als das System (2). Dieser Widerspruch zeigt, daß es modulo p^{μ} keine nichtidentischen, aber wertgleichen Systeme von Linearformen geben kann.

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A note on the best approximation by linear forms of functions

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1. One of the fundamental problems of the theory of approximation seems to be that of determining the worst order of approximation effected by given means over a given class of functions. As far as we know, the first general result of this type is due to A. N. KOLMOGOROV [1]. He considered the class of functions $f(x)$, $0 \leqq x \leqq 1$, having r first derivatives $f'(x), f''(x), \ldots, f^{(r)}(x)$, where $f^{(r)}(x) \in L^2(0, 1)$, and satisfying the periodicity conditions: $f^{(\nu)}(0) = f^{(\nu)}(1)$, $0 \leqq \nu < r$. The result was: *the worst order of L^2-approximation over this class by linear forms of given n functions can be minimized by the first n functions of the trigonometric system, in fact essentially only by these functions.* KOLMOGOROV's method has been developed and applied to several related problems in recent papers by V. M. TIHOMIROV [4] and G. G. LORENTZ [5].

There are two more papers which study similar questions, using another method. The first one [2] is concerned with L^2-approximation by partial sums $s_n(f, x)$ of the development

$$f(x) \sim \sum_n c_n \varphi_n(x), \tag{1. 1}$$

where $\{\varphi_n(x)\}$ is an arbitrary complete orthonormal system, and $f(x)$ belongs to the class of functions of finite variation, or to the class Lip 1. The second paper [3] gets essentially further; it supplies the required lower estimate in case of L^p-approximation ($1 \leqq p \leqq \infty$) within the class of all r times continuously differentiable functions, the approximation means being the Toeplitz means $\sum_{k=1}^{n} \lambda_k c_k \varphi_k(x)$ of (1. 1) satisfying the condition $\sum_{k=1}^{n} \lambda_k^2 \leqq n$. In addition, [3] dispenses with the restriction that $\{\varphi_n(x)\}$ be a complete system.

Let $\mathfrak{K}$ be a subclass of $L^p(0, 1)$. The aim of this note is to give a simple criterion to determine a system $\{\varphi_\nu\}$ with the property that its n-th linear forms approximate in $\mathfrak{K}$ essentially no worse than the n-th linear forms of any other system. Roughly speaking, our theorem provides the best system for linear approximation within the given class.

In § 4 we give two instances to illustrate this theorem.

2. Notation. L^p stands for $L^p(0, 1)$, $2 \leqq p \leqq \infty$; $\| \; \|_p$ denotes the L^p-norm, $\| \; \| = \| \; \|_\infty$ denotes the $C(0, 1)$-norm. Let $\{f_\nu(x)\}$ denote a given system of functions defined and L^p-integrable in [0, 1], the indexing $f_1, f_2, \ldots$ is supposed to be fixed. By a linear form of order $\leqq n$, corresponding to the system, we mean an expression

$$L_n(x) = \sum_{k=1}^{n} a_{nk} f_k(x) \tag{2.1}$$

whose coefficients are real numbers. Put

$$E_n^{(p)}(f; \{f_\nu\}) = \inf \|f - L_n\|_p$$

where the infimum is to be extended over all possible linear forms (2.1) of order $\leqq n$. In case of $p = \infty$ we write simply $E_n = E_n^{(\infty)}$. If $\mathfrak{K}$ is a subclass of L^p, the "worst best approximation" in $\mathfrak{K}$ is defined by

$$E_n^{(p)}(\mathfrak{K}; \{f_\nu\}) = \sup_{f \in \mathfrak{K}} E_n^{(p)}(f; \{f_\nu\}).$$

Finally, given two systems $\{f_\nu^{(1)}\}$ and $\{f_\nu^{(2)}\}$ in L^p, we say that system $\{f_\nu^{(1)}\}$ provides, in $\mathfrak{K}$, no essentially better L^p-approximation than the system $\{f_\nu^{(2)}\}$, if

$$E_n^{(p)}(\mathfrak{K}; \{f_\nu^{(1)}\}) \geqq K_1 \cdot E_n^{(p)}(f; \{f_\nu^{(2)}\}) \qquad (n = 1, 2, \ldots)$$

where $K_1 > 0$ is independent of n. In case of $p = \infty$ we will speak of uniform approximation rather than L^∞-approximation.

3. Lemma. *Let $\{\varphi_\nu\} \in L^p$ be orthonormal over* [0, 1], *further let $\mathfrak{K}$ be a subclass of L^p and n a positive integer. If there exists a positive constant K_2 such that*

$$g_k^{(n)}(x) \overset{\text{def}}{=} K_2 E_n^{(p)}(\mathfrak{K}; \{\varphi_\nu\}) \cdot \varphi_k(x) \in \mathfrak{K} \qquad (k = 1, 2, \ldots, 2n),$$

then, for any orthonormal system $\{\psi_\nu\}$ in L^p,

$$E_n^{(p)}(\mathfrak{K}; \{\psi_\nu\}) \geqq K_3 E_n^{(p)}(\mathfrak{K}; \{\varphi_\nu\}) \tag{3.1}$$

where K_3 is another positive constant.[1])

Proof. Let $s_n(f; \{\psi_\nu\}) = s_n(f; \{\psi_\nu\}; x)$ stand for the n-th partial sum of the ψ_ν-Fourier series of $f \in L^p$. As is well known, among all linear forms (2.1), s_n provides the best approximation to f in the L^2-norm, namely

$$E_n^{(2)}(f; \{\psi_\nu\}) = \|f - s_n(f; \{\psi_\nu\})\|_2 .$$

Putting $f = \varphi_k$, and in view of $\|f\|_2 \leqq \|f - s_n\|_2 + \|s_n\|_2$,

$$1 = \|\varphi_k\|_2 \leqq E_n^{(2)}(\varphi_k; \{\psi_\nu\}) + \left\{ \int_0^1 s_n^2(\varphi_k; \{\psi_\nu\}; x)\, dx \right\}^{\frac{1}{2}},$$

whence summing for $k = 1, 2, \ldots, 2n$,

$$2n \leqq \sum_{k=1}^{2n} E_n^{(2)}(\varphi_k; \{\psi_\nu\}) + \sum_{k=1}^{2n} \left\{ \int_0^1 s_n^2(\varphi_k; \{\psi_\nu\}; x)\, dx \right\}^{\frac{1}{2}}. \tag{3.2}$$

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[1]) If K_2 is an absolute constant then K_3 also is an absolute constant.

Applying the Schwarz inequality to the last sum, and making subsequently use of the orthonormality of ψ_ν's, we get

$$\sum_{k=1}^{2n}\left\{\int_0^1 s_n^2(\varphi_k;\{\psi_\nu\};x)\,dx\right\}^{\frac{1}{2}} \leqq \sqrt{2n}\left\{\sum_{k=1}^{2n}\int_0^1 s_n^2(\varphi_k;\{\psi_\nu\};x)\,dx\right\}^{\frac{1}{2}} =$$

$$= \sqrt{2n}\left\{\sum_{k=1}^{2n}\sum_{j=1}^{n}\left(\int_0^1 \varphi_k(x)\psi_j(x)\,dx\right)^2\right\}^{\frac{1}{2}}.$$

Finally, by Bessel's inequality

$$\sum_{k=1}^{2n}\left(\int_0^1 \varphi_k(x)\psi_j(x)\,dx\right)^2 \leqq \int_0^1 \psi_j^2(x)\,dx = 1,$$

we obtain

$$\sum_{k=1}^{2n}\left\{\int_0^1 s_n^2(\varphi_k;\{\psi_\nu\};x)\,dx\right\}^{\frac{1}{2}} \leqq n\sqrt{2}.$$

Returning to (3.2)

$$\sum_{k=1}^{2n} E_n^{(2)}(\varphi_k;\{\psi_\nu\}) \geqq (2-\sqrt{2})n,$$

and *a fortiori*

$$\max_{1\leqq k\leqq 2n} E_n^{(2)}(\varphi_k;\{\psi_\nu\}) \geqq \frac{2-\sqrt{2}}{2}. \tag{3.3}$$

Replacing φ_k by $g_k^{(n)}=K_2E_n^{(p)}(\mathfrak{K};\{\varphi_\nu\})\cdot\varphi_k$ and nothing that

$$E_n^{(p)}(f;\{\psi_\nu\})\geqq E_n^{(2)}(f;\{\psi_\nu\}) \qquad (p\geqq 2), \tag{3.4}$$

$$E_n^{(p)}(cf;\{\psi_\nu\})=|c|E_n^{(p)}(f;\{\psi_\nu\}), \tag{3.5}$$

we conclude

$$\max_{1\leqq k\leqq 2n} E_n^{(p)}(g_k^{(n)};\{\psi_\nu\}) \geqq \frac{2-\sqrt{2}}{2}K_1\cdot E_n^{(p)}(\mathfrak{K};\{\varphi_\nu\}),$$

and (3.1) follows.

Theorem. *Let $\mathfrak{K}$ be a subclass of L^p and $\{f_\nu\}$ an arbitrary system of functions belonging to L^p. Suppose that there exists a system $\{\varphi_\nu\}\subset L^p$, orthonormal in* [0, 1], *and a positive constant K_4 such that for every $n=1,2,\dots$ functions*

$$g_k^{(n)}(x)=K_4E_n^{(p)}(\mathfrak{K};\{\varphi_\nu\})\cdot\varphi_k(x) \qquad (k=1,2,\dots,2n)$$

belong to $\mathfrak{K}$. Then

$$E_n^{(p)}(\mathfrak{K};\{f_\nu\})\geqq K_5E_n^{(p)}(\mathfrak{K};\{\varphi_\nu\}) \tag{3.6}$$

where K_5 is another positive constant.[2])

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[2]) If K_4 is an absolute constant then K_5 also is an absolute constant.

Corollary. *Let* $\{\varphi_\nu\}$ *and* $\{\psi_\nu\}$ *be orthonormal over* $[0, 1]$. *If for* $k=1, 2, \ldots, 2n$; $n=1, 2, \ldots$

$$K_6 E_n^{(p)}(\mathfrak{K}; \{\varphi_\nu\})\cdot\varphi_k \in \mathfrak{K}$$

and

$$K_7 E_n^{(p)}(\mathfrak{K}; \{\psi_\nu\})\cdot\psi_k \in \mathfrak{K}$$

where K_6, K_7 *are positive constants, then neither of the systems* $\{\varphi_\nu\}$, $\{\psi_\nu\}$ *provides, in* $\mathfrak{K}$, *an essentially better approximation than the other.*

Proof of the theorem. First of all, we observe that without any loss of generality f_ν's can be supposed linearly independent. For if not, we would reject those expressible as linear forms of the preceding ones and consider the new system, say $\{f_\nu^*\}$, whose elements enjoy the required property. The set of linear forms (2. 1) corresponding to the new system contains obviously all linear forms derived from the original one, and consequently

$$E_n^{(p)}(f; \{f_\nu\}) \geqq E_n^{(p)}(f; \{f_\nu^*\}).$$

Hence it is enough to prove (3. 6) for $E_n^{(p)}(f; \{f_\nu^*\})$.

Next we note that f_ν^*'s can be supposed orthonormal. In fact, the familiar Schmidt orthogonalization-process of $\{f_\nu^*\}$ leaves the set (2. 1) of linear forms of order $\leqq n$ — whence also the numbers $E_n^{(p)}(f; \{f_\nu^*\})$ — unchanged.

Thus we shall suppose $\{f_\nu^*\}$ an orthonormal system. Our lemma can be applied and (3. 6) follows.

4. To illustrate the theorem, we consider two special cases:

a) $p=\infty$, $\varphi_1(x)=1$, $\varphi_{2\nu}(x)=\sqrt{2}\cos\pi\nu x$, $\varphi_{2\nu+1}(x)=\sqrt{2}\sin\pi\nu x$, $\mathfrak{K}=\mathfrak{K}_\alpha=$ the class of 1-periodic functions belonging to $\mathrm{Lip}_1\,\alpha$ $(0<\alpha\leqq 1)$ on the whole real axis. As is well known,

$$E_n(f; \{\varphi_\nu\}) \leqq K_8 n^{-\alpha} \quad \text{for} \quad f\in\mathfrak{K}$$

where $K_8>0$ is an absolute constant. Hence

$$E_n(\mathfrak{K}; \{\varphi_\nu\}) \leqq K_8 n^{-\alpha}.$$

It is easy to see that for any x', x'' and $k=1, 2, \ldots, 2n$

$$\frac{|\varphi_k(x'')-\varphi_k(x')|}{n^\alpha\sqrt{2}} \leqq \begin{cases} \pi k|x''-x'|n^{-\alpha} \\ 2n^{-\alpha} \end{cases} \leqq 2\pi|x''-x'|^\alpha,$$

so that

$$g_k^{(n)}(x) = 2^{-\frac{3}{2}}\pi^{-1}K_8^{-1}E_n(\mathfrak{K}_\alpha; \{\varphi_\nu\})\cdot\varphi_k(x)\in\mathfrak{K} \qquad (k=1, 2, \ldots, 2n).$$

This means: *No system* $\{f_\nu\}$ *provides, in* $\mathfrak{K}_\alpha$, *an essentially better uniform approximation than the trigonometric system.*

b) $p=\infty$, $\{\varphi_\nu\}$ the same as in a), $\mathfrak{K}=\mathfrak{K}^{(r)}=$ the class of 1-periodic functions f whose r-th derivative is continuous in $(-\infty, +\infty)$ and $\max\limits_{0\leqq l\leqq r}\|f^{(l)}\|\leqq 1$. By a well-known theorem

$$E_n(\mathfrak{K}^{(r)}; \{\varphi_\nu\}) \leqq K_9(r) n^{-r}.$$

Since for $0 \leqq l \leqq r$, $k=1, 2, \ldots, 2n$

$$\left|2^{-\frac{1}{2}} n^{-r} \varphi_k^{(l)}(x)\right| \leqq \left(\frac{k\pi}{n}\right)^r \leqq (2\pi)^r,$$

it follows that

$$g_k^{(n)}(x) = 2^{-r-\frac{1}{2}} \pi^{-r} (K_9(r))^{-1} E_n(\mathfrak{K}^{(r)}; \{\varphi_\nu\}) \cdot \varphi_k(x) \in \mathfrak{K}^{(r)} \qquad (k = 1, 2, \ldots, 2n).$$

Hence: *No system $\{f_\nu\}$ provides, in $\mathfrak{K}^{(r)}$, an essentially better uniform approximation than the trigonometric system.*

Remark. The statements about the examples a) and b) are not new. But, if we take into account that the classes discussed, and many others, contain, besides the trigonometric system, also functions $g_k^{(n)}(x)$ of some other orthonormal systems, e. g. those of certain Sturm—Liouville systems, then, by our corollary we can show that all these systems provide, in the corresponding classes, essentially the same best uniform approximation; a result the direct proof of which would be rather lengthy.

5. It is obvious that our theorem and its consequences remain true, except for the constants, if we replace [0, 1] by an arbitrary finite interval $[a, b]$, and understand by "orthonormal system" a system of functions orthonormal, in $[a, b]$, relatively to a weight function $w(x) \geqq 0$.

Denote by $\{p_n(x)\}$ the system of orthonormal polynomials determined by the weight function $w(x)$. By our theorem it follows that, if the class $\mathfrak{K}$ contains the functions

$$g_k^{(n)}(x) = K_{10} E_n(\mathfrak{K}; \{p_\nu\}) \cdot p_k(x) \qquad (k=1, 2, \ldots, 2n;\ n=1, 2, \ldots),$$

we get for no system $\{f_\nu\} \subset C(a, b)$ an essentially better uniform approximation in $\mathfrak{K}$ than the best one provided by polynomials. Therefore, denoting by $\mathfrak{A}$ the class of all analytical funtions $f(x)$ possessing derivatives $\|f^{(r)}(x)\| \leqq 1$, $r=0, 1, \ldots$ the essentially best uniform approximation in $\mathfrak{A}$ is provided by the system of all polynomials. This result is reversible in a certain sense:

If there exists a constant $K_{11} > 0$ such that for any $f \in \mathfrak{A}$ we have

$$E(f; \{f_\nu\}) \leqq K_{11} \|f^{(r)}\| \qquad (r=0, 1, \ldots),$$

then the set $\{L_n(x)\}$ of all linear forms corresponding to the system $\{f_\nu(x)\}$ contains all polynomials.

Proof. Put $P_k(x)$ a polynomial of degree k having the norm $\|P_k\| \leqq 1$ in $[a, b]$, and consider the functions

$$\psi_k(x) = \frac{P_k(x)}{K_{12} \cdot k^{2k}}$$

where K_{12} is a suitably chosen positive constant. The functions $\psi_k(x)$ belong to the class $\mathfrak{A}$. Indeed, taking $r \leqq k$, by a well-known inequality of Markov—Bernstein, we obtain $\|P_k^{(r)}\| \leqq K_{12} k^{2k} \|P_k\|$ where K_{12} depends only on r and the lenght of $[a, b]$. Hence

$$\|\psi_k^{(r)}\| \leqq \frac{\|P_k^{(r)}\|}{K_{12} k^{2k}} \leqq \|P_k\| \leqq 1.$$

If $r>k$, we have evidently $\|\psi_k^{(r)}\|=0$. Fix, now, $r>k$. Since there exists at least one linear form $L_n^*(x)$ corresponding to $\{f_\nu(x)\}$ such that $\|\psi_k - L_n^*\| = E_n(\psi_k; \{f_\nu\})$, it follows

$$\|\psi_k - L_n^*\| = E_n(\psi_k; \{f_\nu\}) \leqq K_{11} \cdot \|\psi_k^{(r)}\| = 0,$$

and therefore $L_n^*(x) \equiv \psi_k(x)$, which proves our statement.

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Über die Divergenz der Partialsummen von Orthogonalreihen

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Einleitung

Sei $\{a_n\}$ eine gegebene Folge von reellen Zahlen und sei $(0=)m_0<m_1<\ldots<m_i<\ldots$ eine gegebene Indexfolge. Wir setzen

$$A_i=\{a_{m_{i-1}+1}^2+\ldots+a_{m_i}^2\}^{1/2} \qquad (i=1,2,\ldots).$$

Wir beweisen den folgenden

Satz I. *Ist* $A_n \geqq A_{n+1}$ $(n=1,2,\ldots)$ *und*

$$\sum_{n=2}^{\infty} A_n^2 \log^2 n = \infty, \tag{1}$$

so kann ein in $[0,1]$ *orthonormiertes, gleichmäßig beschränktes Funktionensystem* $\{\psi_n(x)\}$ $(|\psi_n(x)|\leqq K;\ n=1,2,\ldots;\ x\in[0,1])$ *derart angegeben werden, daß die* m_i*-ten Partialsummen der Reihe*

$$\sum_{n=0}^{\infty} a_n\psi_n(x) \tag{2}$$

im Intervall $[0,1]$ *fast überall divergieren.*

K. TANDORI [4] (Satz 3) hat diesen Satz *ohne* die Forderung der gleichmäßigen Beschränktheit bewiesen. Von ihm stammt auch das Problem, ob der Satz auch in der obigen schärferen Form richtig ist?

Bezeichne $N(p_n)$ ein beliebiges Summationsverfahren mit der Eigenschaft, daß die Konvergenz fast überall der p_n-ten Partialsummen notwendig dafür ist, daß jede Orthogonalreihe $\sum_{n=0}^{\infty} a_n\varphi_n(x)$ mit $\sum a_n^2<\infty$ fast überall $N(p_n)$-summierbar sei. Nach einem Satz von S. KACZMARZ ([1], Satz 5.7.4) ist jedes permanente Toeplitzsche Summationsverfahren ein $N(p_n)$-Summationsverfahren mit irgendeiner Folge $\{p_n\}$.

Offenbar hat Satz I die

Folgerung I. *Sei* $\{p_n\}$ *die zur* $N(p_n)$*-Summierbarkeit zugehörige Indexfolge. Ist*

$$A_i^2(p) = \sum_{n=p_i+1}^{p_{i+1}} a_n^2 \geqq \sum_{n=p_{i+1}+1}^{p_{i+2}} a_n^2 = A_{i+1}^2(p)$$

und

$$\sum_{n=2}^{\infty} A_n^2(p) \log^2 n = \infty,$$

so kann ein gleichmäßig beschränktes orthonomiertes Funktionensystem $\{\psi_n(x)\}$ *in* $[0, 1]$ *angegeben werden, für welches die Reihe* (2) *fast überall nicht* $N(p_n)$*-summierbar ist.*

Einer der Verfasser [2] hat bewiesen, daß die Behauptungen, die die Existenz eines orthonormierten Funktionensystems mit gewissen Divergenzeigenschaften behaupten, derart verschärft werden können, daß man für die orthonormierten Funktionensysteme mit den betreffenden Divergenzeigenschaften auch Polynomsysteme wählen kann.

Durch Anwendung des folgenden Satzes ([2], Satz II) können wir den Satz I ebenso verschärfen:

Es seien vorgegeben: eine reelle Zahlenfolge $\{s_n\}$, ein im Intervall $[0, 1]$ orthonormiertes Funktionensystem $\{\varphi_n(x)\}$, eine Folge von meßbaren Teilmengen E_m von $[0, 1]$, eine Indexfolge $\{\nu_m\}$ $(0 = \nu_0 < \nu_1 < \ldots < \nu_m < \ldots)$ und eine positive Zahl ε. Wir nehmen an, daß $\mu(\overline{\lim_{m\to\infty}} E_m) = 1$ ist, und daß es für jedes $x \in E_m$ einen Index $\mu_m(x)$ $(< \nu_{m+1} - \nu_m)$ mit

$$|s_{\nu_m+1}\varphi_{\nu_m+1}(x) + \ldots + s_{\nu_m+\mu_m(x)}\varphi_{\nu_m+\mu_m(x)}(x)| \geqq D(m)$$

gibt, wobei $\{D(m)\}$ eine positive, monoton nichtabnehmende Zahlenfolge ist.

Dann kann ein in $[0, 1]$ orthonormiertes Polynomsystem $\{P_n(x)\}$ angegeben werden, derart, daß die Ungleichung

$$|s_{\nu_m+1}P_{\nu_m+1}(x) + \ldots + s_{\nu_m+\mu_m(x)}P_{\nu_m+\mu_m(x)}(x)| \geqq (1-\varepsilon)D(m)$$

für fast alle $x \in [0, 1]$ bei unendlich vielen Werten von m erfüllt wird. Ist das Funktionensystem $\{\varphi_n(x)\}$ in $[0, 1]$ glechmäßig beschränkt, so kann auch das Polynomsystem $\{P_n(x)\}$ gleichmäßig beschränkt gewählt werden.

Mit Zuhilfenahme der Beweisführung des Satzes I, durch Anwendung des zitierten Satzes ergibt sich:

Satz II. *Der Satz I läßt sich so verschärfen, daß das betreffende Orthonormalsystem* $\{\psi_n(x)\}$ *aus Polynomen besteht.*

§ 1. Hilfssätze

Zum Beweis unseres Satzes benötigen wir die folgenden Hilfssätze:

Hilfssatz I. *Ist* $\sum_{n=0}^{\infty} c_n^2 < \infty$ *und*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \operatorname{sign} \sin 2^{n+1}\pi x \equiv \sum_{n=0}^{\infty} c_n r_n(x) \qquad (0 \leqq x \leqq 1);$$

so gilt

$$A\left(\sum_{n=0}^{\infty} c_n^2\right)^{1/2} \leqq \int_0^1 |f(x)|\, dx \leqq B\left(\sum_{n=0}^{\infty} c_n^2\right)^{1/2},$$

wobei A und B positive absolute Konstanten bedeuten.

Siehe z. B. [5] S. 213.

Hilfssatz II. *Sei $\{c_n\}$ eine gegebene Koeffizientenfolge. Bezeichnet $E_{n,m}$ die Menge der Punkte, für die die Ungleichung*

$$\left|\sum_{\nu=n}^{n+m} c_\nu r_\nu(x)\right| > \frac{A}{2}\left\{\sum_{\nu=n}^{n+m} c_\nu^2\right\}^{1/2}$$

erfüllt ist, so sind diese Mengen $E_{n,m}$ für jede n und m einfach[1]) *und es gilt*

$$\mu(E_{n,m}) \geqq \frac{A^2}{4}, \tag{1. 1}$$

wobei A und $r_n(x)$ wie im Hilfssatz I definiert sind.

Beweis. Nach dem Hilfssatz I gilt

$$A\left\{\sum_{\nu=n}^{n+m} c_\nu^2\right\}^{1/2} \leqq \int_0^1 \left|\sum_{\nu=n}^{n+m} c_\nu r_\nu(x)\right| dx = \left(\int_{CE_{n,m}} + \int_{E_{n,m}}\right)\left|\sum_{\nu=n}^{n+m} c_\nu r_\nu(x)\right| dx \leqq {}^{2})$$

$$\leqq \frac{A}{2}\left\{\sum_{\nu=n}^{n+m} c_\nu^2\right\}^{1/2} + \left\{\mu(E_{n,m}) \sum_{\nu=n}^{n+m} c_\nu^2\right\}^{1/2}.$$

Daraus ergibt sich die Behauptung (1. 1). Nach der Definition der Rademacherschen Funktionen ist klar, daß die Mengen $E_{n,m}$ einfach sind.

Damit ist der Beweis fertig.

Hilfssatz III. *Sei $\{c_n\}$ eine positive, monoton nichtzunehmende Zahlenfolge, die die Bedingung*

$$\sum_{n=2}^{\infty} c_n^2 \log^2 n = \infty$$

erfüllt. Dann kann eine Indexfolge $N_0 < N_1 < \ldots < N_m < \ldots$ ($N_0 = 0$, $N_1 > 4$), ein im Intervall [0, 1] *orthonormiertes und gleichmäßig beschränktes Funktionensystem $\{g_n(x)\}$ und eine Folge von einfachen Mengen G_m ($\subseteqq [0, 1]$) angegeben werden, derart, daß die folgenden Bedingungen erfüllt sind:*

---

[1]) Eine Punktmenge nennen wir einfach, wenn sie die Vereinigungsmenge endlich vieler Intervalle ist. Mit $\mu(H)$ wird das Lebesguesche Maß der Menge H bezeichnet.

[2]) CH bezeichnet die Komplementärmenge der Menge H in bezug auf das Intervall [0, 1].

a) *zu jedem* $x \in G_m$ *gibt es eine natürliche Zahl* $n_m(x)$ $(<N_{m+1}-N_m)$, *für die Funktionswerte* $g_{N_m}(x), \ldots, g_{N_m+n_m(x)}(x)$ *gleiches Verzeichen haben und*

$$\left|\sum_{i=N_m}^{N_m+n_m(x)} c_i g_i(x)\right| > D > 0$$

mit einer von m *und* x *unabhängigen positiven Zahl* D *gilt,*

b) *es gilt*

$$\sum_{m=0}^{\infty} \mu(G_m) = \infty. \tag{1.2}$$

Dies folgt aus dem Hilfssatz II von K. TANDORI [3] (s. den Beweis des Hilfssatzes III in [3]).

§ 2. Beweis des Satzes I

Nach der Bedingung (1) des Satzes I kann der Hilfssatz III für die Folge $\{A_n\}$ angewandt werden; die entsprechenden Funktionen, Mengen und Indexfolge bezeichnen wir mit $g_n(x)$, G_m und $\{N_m\}$;

$$|g_n(x)| \leqq K \ (n=0, 1, \ldots), \ G_m \subseteqq [0, 1] \ (m=0, 1, \ldots).$$

Sei r eine feste nicht-negative ganze Zahl und wir setzen

$$\mu_r = \max_{N_r \leqq i < N_{r+1}} (m_i - m_{i-1}) \qquad (m_{-1} = 0).$$

Wir teilen das Interwall [0, 1] in $Q_r = 2^{\mu_r + [2^4 A^{-2}]}$ [3]) Teilintervalle gleicher Länge $I_q = [u_q, v_q]$ $(1 \leqq q \leqq Q_r)$ ein. Sei $N_r \leqq i < N_{r+1}$ und

$$g_{i_q}(x) = \begin{cases} g_i\left(\dfrac{x-u_q}{v_q-u_q}\right) & \text{für } u_q < x < v_q, \\ 0 & \text{sonst.} \end{cases}$$

Es gilt

$$\int_{I_q} g_{l_q}(x) g_{k_q}(x)\, dx = \begin{cases} 0 & \text{für } l \neq k, \\ 1/Q_r & \text{für } l = k. \end{cases} \tag{2.1}$$

Mit Rücksicht darauf, daß für jedes n und x $(0 \leqq x \leqq \frac{1}{2})$ $r_n(\frac{1}{2}-x) = -r_n(\frac{1}{2}+x)$ gilt, folgt aus dem Hilfssatz II, daß die Gesamtlänge der Teilintervalle I_q, auf den die Ungleichung

$$\sum_{n=m_{i-1}+1}^{m_i} a_n r_{n-m_{i-1}}(x) > \frac{A}{2}\left\{\sum_{n=m_{i-1}+1}^{m_i} a_n^2\right\}^{1/2} \tag{2.2}$$

gilt, für jedes $i = N_r, \ldots, N_{r+1}-1$ größer als $2^{-3}A^2$ ist. Also ist die Zahl der Teilintervalle I_q, auf den die Ungleichung (2.2) gilt, größer als $Q_r 2^{-3} A^2 - 1 = p_r$. Sei i $(N_r \leqq i < N_{r+1})$ eine feste natürliche Zahl. Zu jedem Intervall I_q mit $q < p_r$, auf

[3]) $[\alpha]$ bezeichnet den ganzen Teil von α.

dem (2. 2) nicht erfüllt ist, nehmen wir ein Intervall $I_{q'}$ mit $q' > p_r$; auf dem (2. 2) erfüllt ist, und zwar so, daß verschiedenen I_q verschiedene $I_{q'}$ zugeordnet werden. Wir vertauschen die Werte der Funktionen $r_j(x)$ $(j=1, \dots, m_i - m_{i-1})$ auf den einander zugeordneten Intervallen I_q, $I_{q'}$. Die so erhaltenen Funktionen bezeichnen wir mit $r_{ij}(x)$ $(j=1, 2, \dots, m_i - m_{i-1})$. Diese Umordnung machen wir für jedes i $(N_r \leqq i < N_{r+1})$. Nach den obigen besteht für jedes $x \in I_q$ $(q \leqq p_r)$

$$\sum_{n=m_{i-1}+1}^{m_i} a_n r_{i,n-m_{i-1}}(x) > \frac{A}{2} \left\{ \sum_{n=m_{i-1}+1}^{m_i} a_n^2 \right\}^{1/2}. \tag{2. 3}$$

Wir setzen

$$\gamma_{m_{i-1}+j}(x) = \sum_{q=1}^{Q_r} r_{ij}(x) g_{iq}(x) \qquad (N_r \leqq i < N_{r+1};\ j=1, \dots, m_i - m_{i-1}).$$

Mit einer einfachen Rechnung können wir einsehen, daß das so definierte System $\gamma_k(x)$ $(m_{N_r-1} < k \leqq m_{N_{r+1}-1},\ m_{-1} = 0)$ orthonormiert ist. Sei $l = m_{i-1} + j$. Nach (2.1) gilt

$$\int_0^1 \gamma_l^2(x)\,dx = \sum_{q=1}^{Q_r} \int_{I_q} r_{ij}^2(x) g_{iq}^2(x)\,dx = \sum_{q=1}^{Q_r} \int_{I_q} g_{iq}^2(x)\,dx = Q_r \frac{1}{Q_r} = 1,$$

d.h. diese Funktionen sind normiert.

Ist $l_1 \neq l_2$ und $l_1 = m_{i_1-1} + j_1$, $l_2 = m_{i_2-1} + j_2$, so ist entweder $i_1 \neq i_2$, oder $i_1 = i_2$ und $j_1 \neq j_2$. Da die Funktionen $r_{ij}(x)$ $(i = N_r, \dots, N_{r+1}-1; j = 1, \dots, m_i - m_{i-1})$ in allen Intervallen I_q $(q = 1, \dots, Q_r)$ konstant sind, folgt aus (2. 1) für $i_1 \neq i_2$

$$\int_0^1 \gamma_{l_1}(x)\gamma_{l_2}(x)\,dx = \sum_{q=1}^{Q_r} \int_{I_q} r_{i_1 j_1}(x) r_{i_2 j_2}(x) g_{i_1 q}(x) g_{i_2 q}(x)\,dx =$$

$$= \sum_{q=1}^{Q_r} \pm \int_{I_q} g_{i_1 q}(x) g_{i_2 q}(x)\,dx = 0,$$

weiterhin für $i_1 = i_2 = i$, $j_1 \neq j_2$

$$\int_0^1 \gamma_{l_1}(x)\gamma_{l_2}(x)\,dx = \sum_{q=1}^{Q_r} \int_{I_q} r_{ij_1}(x) r_{ij_2}(x) g_{iq}^2(x)\,dx = \sum_{q=1}^{Q_r} \operatorname*{sign}_{x \in I_q} r_{ij_1}(x) r_{ij_2}(x) \frac{1}{Q_r} =$$

$$= \sum_{q=1}^{Q_r} \int_{I_q} r_{ij_1}(x) r_{ij_2}(x)\,dx = \int_0^1 r_{j_1}(x) r_{j_2}(x)\,dx = 0.$$

Also ist das System auch orthogonal.

Sei

$$\bar{G}_r = \sum_{q=1}^{p_r} G_r(I_q),$$

wobei $E(I)$ das mit der linearen Transformation $x=(v-u)y+u$ erhaltene Bild in $I=[u, v]$ der Menge $E(\subseteqq[0, 1])$ bedeutet. Es ist klar, daß die Menge $\bar{G}_r$ einfach ist, und daß

$$\mu(\bar{G}_r) = \sum_{q=1}^{p_r} \mu(G_r(I_q)) = \sum_{q=1}^{p_r} \frac{1}{Q_r} \mu(G_r)$$

gilt. Nach der Definition von p_r folgt hieraus

$$\mu(\bar{G}_r) \geqq 2^{-4} A^2 \mu(G_r). \tag{2. 4}$$

Auf Grund der Definition von $\gamma_k(x)$ folgt, daß $|\gamma_k(x)| \leqq K$. Ist endlich $x \in \bar{G}_r$, d.h. ist $x \in G_r(I_{q_0})$ mit einem $q_0(<p_r)$, so gilt nach (2.2) und nach dem Hilfssatz III (da $g_{N_r}(x), \ldots, g_{N_r+n_r(x)}(x)$ gleiches Vorzeichen haben) die Ungleichung

$$\left|\sum_{i=N_r}^{N_r+n_r(x)} \sum_{k=m_{i-1}+1}^{m_i} a_k \gamma_k(x)\right| = \left|\sum_{i=N_r}^{N_r+n_r(x)} \sum_{k=m_{i-1}+1}^{m_i} a_k r_{i,k-m_{i-1}}(x) g_{iq_0}(x)\right| = \tag{2. 5}$$

$$= \left|\sum_{i=N_r}^{N_r+n_r(x)} g_{iq_0}(x) \sum_{k=m_{i-1}+1}^{m_i} a_k r_{i,k-m_{i-1}}(x)\right| \geqq \left|\sum_{i=N_r}^{N_r+n_r(x)} g_{iq_0}(x) \frac{A}{2} A_i\right| \geqq \frac{AD}{2}.$$

Nach dem obigen können wir leicht ein Funktionensystem $\{\psi_n(x)\}$ mit $|\psi_n(x)| \leqq K$, und eine Folge von einfachen Mengen H_k definieren, für die die folgenden Bedingungen erfüllt sind:

a) für jedes $x \in H_k$ gilt

$$\left|\sum_{i=N_k}^{N_k+n_k(x)} \sum_{l=m_{i-1}+1}^{m_i} a_l \psi_l(x)\right| \geqq \frac{AD}{2}, \tag{2. 6}$$

b) die Mengen H_k $(k=0, 1, \ldots)$ sind stochastisch unabhängig und

$$\sum_{k=0}^{\infty} \mu(H_k) = \infty.$$

Sei $\psi_k(x) = \gamma_k(x)$ für $k=1, \ldots, m_{N_1-1}$ und $H_0 = \bar{G}_0$. Nach der Definition und nach (2. 5) ist es klar, daß die Behauptung a) für $k=0$ erfüllt ist.

Sei nun $\nu(\geqq 1)$ eine beliebige natürliche Zahl. Wir nehmen an, daß die Treppenfunktionen $\psi_k(x)$ $(k=1, 2, \ldots, m_{N_\nu-1})$ und die Mengen H_l $(l=1, \ldots, \nu)$ schon definiert sind; die $\psi_k(x)$ bilden im Intervall $[0, 1]$ ein orthonormiertes System, es gilt $|\psi_k(x)| \leqq K$, ferner ist die Bedingung a) für $k=0, \ldots, \nu$ erfüllt; die Mengen H_l $(l \leqq \nu)$ sind stochastisch unabhängig und es gilt

$$\sum_{l=1}^{\nu} \mu(H_l) = \sum_{l=1}^{\nu} \mu(\bar{G}_l). \tag{2. 7}$$

Dann kann man das Intervall $[0, 1]$ in endlich viele Teilintervalle I_ϱ $(\varrho=1, \ldots, R)$ zerlegen, derart, daß in den einzelnen Teilintervallen die Funktionen $\psi_k(x)$ $(k=0, 1, \ldots$ $\ldots, m_{N_\nu-1})$ konstant sind. Wir bezeichnen mit I_ϱ^1 und I_ϱ^2 die zwei Hälften des Intervalls I_ϱ $(\varrho=1, 2, \ldots, R)$ und wir setzen

$$\psi_k(x) = \sum_{\varrho=1}^{R} \left(\gamma_k(I_\varrho^1; x) - \gamma_k(I_\varrho^2; x)\right)^{4)}$$

für $k = m_{N_\nu}, \ldots, m_{N_{\nu+1}-1}$ und

$$H_{\nu+1} = \bigcup_{\varrho=1}^{R} \left(\bar{G}_{\nu+1}(I_\varrho^1) \cup \bar{G}_{\nu+1}(I_\varrho^2)\right).$$

Auf Grund dieser Definition ist es klar, daß die Funktionen $\psi_k(x)$ $(k=1, \ldots, m_{N_{\nu+1}-1})$ in $[0, 1]$ ein orthonormiertes System bilden, weiterhin

$$\mu(H_{\nu+1}) = \mu(\bar{G}_{\nu+1}), \tag{2. 8}$$

und die Mengen $H_1, \ldots, H_{\nu+1}$ stochastisch unabhängig sind. Nach (2. 8) ist klar, daß (2. 7) für $\nu+1$ erfüllt ist.

Ist $x \in H_{\nu+1}$, d.h. $x \in G_{\nu+1}(I_{\varrho_0}^i)$ für ein ϱ_0 und für $i=1$ oder 2, so besteht nach (2. 5)

$$\left| \sum_{i=N_\nu+1}^{N_{\nu+1}+n_{\nu+1}(x)} \sum_{k=m_{i-1}+1}^{m_i} a_k \psi_k(x) \right| = \left| \sum_{i=N_\nu+1}^{N_{\nu+1}+n_{\nu+1}(x)} \sum_{k=N\nu+1}^{m_i} a_k \gamma_k(I_{\varrho_0}^i; x) \right| \geqq \frac{AD}{2}.$$

Durch vollständige Induktion erhalten wir also ein im Intervall [0, 1] orthonormiertes Funktionensystem $\{\psi_k(x)\}$ $(k=0, 1, \ldots)$ und Mengen H_k $(k=1, \ldots)$, derart, daß die Behauptungen a) und b) erfüllt sind; nach (1. 2), (2. 4) und (2. 7) ist nämlich $\sum_{k=1}^{\infty} \mu(H_k)$ unendlich.

Auf Grund des Borel—Cantellischen Lemmas ergibt sich aus der Behauptung b):

$$\mu(\overline{\lim_{k\to\infty}} H_k) = 1,$$

d.h. $x \in \overline{\lim_{k\to\infty}} H_k$ für fast alle x. Für $x \in \overline{\lim_{k\to\infty}} H_k$ gilt aber die Ungleichung (2. 6) für unendlich viele k.

Damit haben wir den Satz I bewiesen.

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4) Ist $I=[u, v]$ ein beliebiges Intervall, so definieren wir:

$$f(I; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{für } u < x < v, \\ 0 & \text{sonst.} \end{cases}$$

Bemerkung zu meiner Arbeit: „Über Konvergenz- und Summationseigenschaften von Haarschen Reihen"*)

Von L. LEINDLER in Szeged

Herr Professor P. L. ULJANOV hat mich darauf aufmerksam gemacht, daß mein Beweis für Satz I in der genannten Arbeit lückenhaft ist. Nämlich ist die folgende Behauptung auf S. 21 falsch: „Man kann leicht einsehen, daß das System $\{2^{-[\log k]/2}\chi_k(x)\chi_l(x)\}$ mit $l\in I(k,\infty)$ in $(0,1)$ orthonormiert ist." Es erübrigt sich, meinen Beweis zu vervollständigen, da für einen allgemeineren Satz ein Beweis vorliegt, s. P. L. ULJANOV, О рядах по системе Хаара с монотонными коэффициентами, *Изв. Ан. СССР*, **28** (1964), 925—950.

*) *Acta Sci. Math.*, **26** (1965), 19—30.

Homomorphisms of certain commutative lattice ordered semigroups

By P. J. McCARTHY in Lawrence (Kansas, U. S. A.)*

Let S be a semigroup. It is well-known that a homomorphic image of S is, to within isomorphism, of the form S/θ where θ is a congruence on S. In this note we shall show that if S is a commutative lattice ordered semigroup [2, Chapter XIII], with certain additional properties, then only congruences of a certain type are required to describe all of the homomorphic images of S. Then we shall point out a particularly interesting example of a class of semigroups which have all of these properties. In this note, when we refer to a homomorphism from one partially ordered semigroup into another we shall always mean one that preserves ordering.

First, assume that S is a commutative partially ordered semigroup, so that there is a partial ordering on S with the property that if $a, b, c \in S$ and if $a \leqq b$ then $ac \leqq bc$. Also assume that S has an identity e such that $a \leqq e$ for all $a \in S$.

Let M be a subsemigroup of S. For each $a \in S$ we set $a' = \{x \in S | mx \leqq a$ for some $m \in M\}$. Since $m \leqq e$ for all $m \in M$ we have $ma \leqq a$ and so $a \in a'$ for all $a \in S$. We define a relation θ on S by $a \equiv b(\theta)$ if and only if $a' = b'$. This is an equivalence relation on S and we easily verify the following facts:

(1) if $a \in b'$ then $a' \subseteqq b'$,

(2) if $a \equiv b(\theta)$ then $ac \equiv bc(\theta)$ for all $c \in S$,

(3) if $a \leqq c \leqq b$ and $a \equiv b(\theta)$ then $a \equiv c(\theta)$.

Thus, θ is a congruence on S and we can consider the semigroup S/θ. We denote the equivalence class of $a \in S$ with respect to θ by $\theta(a)$, and we also denote the natural homomorphism from S onto S/θ by θ.

From now on we shall assume that S is a lattice with respect to its partial ordering and that $a(b \vee c) = ab \vee ac$ for all $a, b, c \in S$.

(4) if $a \equiv b(\theta)$ we have $a \vee c \equiv b \vee c(\theta)$ and $a \wedge c \equiv b \wedge c(\theta)$ for all $c \in S$.

For, since $a \in b'$ there is an $m \in M$ such that $ma \leqq b$. Then $m(a \vee c) = ma \vee mc \leqq$ $\leqq b \vee c$ and $m(a \wedge c) \leqq ma \wedge mc \leqq b \wedge c$. Hence $a \vee c \in (b \vee c)'$ and $a \wedge c \in (b \wedge c)'$, and so $(a \vee c)' \subseteqq (b \vee c)'$ and $(a \wedge c)' \subseteqq (b \wedge c)'$. By symmetry, $(b \vee c)' \subseteqq (a \vee c)'$ and $(b \wedge c)' \subseteqq (a \wedge c)'$.

*) Supported in part by NSF GP-1738.

We can therefore define operations $\vee$ and $\wedge$ on S/θ by $\theta(a) \vee \theta(b) = \theta(a \vee b)$ and $\theta(a) \wedge \theta(b) = \theta(a \wedge b)$, and with respect to these operations, S/θ is a lattice ordered semigroup. It is quite trivial that the properties required of the operations $\vee$ and $\wedge$ hold. It remains to verify that the ordering induced on S/θ by the lattice structure is compatible with the multiplication on S/θ. We have $\theta(a) \leqq \theta(b)$ if and only if $\theta(b) = \theta(a) \vee \theta(b) = \theta(a \vee b)$, and so if $\theta(a) \leqq \theta(b)$ we have for all $\theta(c) \in S/\theta$, $\theta(bc) = \theta(b)\theta(c) = \theta(a \vee b)\theta(c) = \theta((a \vee b)c) = \theta(ac \vee bc)$. Hence $\theta(a)\theta(c) \leqq \theta(b)\theta(c)$. Note that $\theta(e)$ is the identity of S/θ and that $\theta(a) \leqq \theta(e)$ for all $\theta(a) \in S/\theta$.

Now consider a homomorphism h from S onto a partially ordered semigroup T. If we set $M = \{m \in S | h(m) = h(e)\}$ then M is a subsemigroup of S. Let θ be the congruence on S associated with M in the manner we have described. If $a \in S$ we set $f\theta(a) = h(a)$. If we show that f is a well-defined mapping from S/θ into T, then it is clear that f is a homomorphism from S/θ onto T such that $f\theta = h$. Suppose that $\theta(a) = \theta(b)$, i.e., $a \equiv b(\theta)$. Then there are elements $m, n \in M$ such that $ma \leqq b$ and $nb \leqq a$. Hence $nma \leqq nb \leqq a$ and so $h(a) = h(mna) \leqq h(nb) = h(b) \leqq h(a)$. Thus $h(a) = h(b)$ and we conclude that f is well-defined.

We seek conditions under which f will be an isomorphism. A suitable condition, for our purposes, is that both S and T be residuated [2, p. 189] and that h preserve residuals. For, suppose that this is the case, and that $h(a) = h(b)$. Then $h(e)h(b) = h(a)$ and so $h(e) \leqq h(a):h(b)$. Hence $h(e) = h(a):h(b) = h(a:b)$, which means that $a:b \in M$. Since $(a:b)b \leqq a$ we have $b' \subseteqq a'$. Similarly $a' \subseteqq b'$ and therefore $a \equiv b(\theta)$ and $\theta(a) = \theta(b)$. We can summarize all of this as the

Theorem. *Let S be a commutative residuated lattice ordered semigroup with an identity e such that $a \leqq e$ for all $a \in S$. Let T be a residuated partially ordered semigroup and suppose there is a homomorphism h from S onto T which preserves residuals. Then there is a subsemigroup M of S such that if θ is the congruence on S determined as above by M, then there is an isomorphism f from S/θ onto T such that $f\theta = h$.*

Remark 1. If S and T are as in the statement of the theorem, then T becomes a lattice ordered semigroup when we define meet and join on T by $h(a) \wedge h(b) = h(a \wedge b)$ and $h(a) \vee h(b) = h(a \vee b)$.

Remark 2. Let S be as in the statement of the theorem, let M be a subsemigroup of S, and let θ be the congruence on S determined by M. Then the semigroup S/θ is residuated and the homomorphism θ preserves residuals. To show this we shall verify that $\theta(a:b)$ is the residual of $\theta(a)$ by $\theta(b)$. We have $\theta(a:b)\theta(b) = \theta((a:b)b) \leqq \theta(a)$. Furthermore, suppose that $\theta(c)\theta(b) \leqq \theta(a)$. Then $\theta(cb) \leqq \theta(a)$ and so $\theta(cb) = \theta(a) \wedge \theta(cb) = \theta(a \wedge cb)$. Thus, for some $m \in M$, $mcb \leqq a \wedge cb \leqq a$. Hence $mc \leqq a:b$ and so $\theta(mc) \leqq \theta(a:b)$. Since $\theta(m) = \theta(e)$, as is easily seen, $\theta(mc) = \theta(m)\theta(c) = \theta(c)$. Hence $\theta(c) \leqq \theta(a:b)$. Therefore, $\theta(a:b)$ is the required residual [2, p. 189]. More generally, these conditions are satisfied by the residuated multiplicative lattices, which have been studied by Ward and Dilworth (see [1] and the references at the end of that paper).

Let L be a residuated multiplicative lattice: then L is a commutative residuated lattice ordered semigroup with an identity I such that $A \leqq I$ for all $A \in L$. The formation of the multiplicative lattice L/θ, where θ is determined as above by a subsemigroup (i.e., multiplicatively closed set) of L, is an abstract construction of the lattice of ideals of a ring of quotients of a commutative ring with identity. A special case of this construction was discussed by DILWORTH [1, pp. 489—491]. Let L be a Noether lattice and let $D \in L$. If M is the set of all $A \in L$ such that D is not greater than or equal to any of the primes associated with a normal decomposition of A, then M is a subsemigroup of L, and if θ is the congruence on L determined as above by M, then L/θ is precisely the congruence lattice L_D of Dilworth. In particularl if D is a prime P of L then $A \in M$ if and only if $A \not\leqq P$, for if $A \leqq P$ then some minima, prime associated with a normal decomposition of A must be less than or equal to P.

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On $|C, 1|_k$ summability factors of infinite series

By S. M. MAZHAR in Aligarh (India)

1. Let Σa_n be a given infinite series with partial sums s_n, and let $t_n = t_n^0 = na_n$. By σ_n^α and t_n^α we denote the n-th Cesàro means of order α $(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{t_n\}$, respectively. The series Σa_n is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ $(k \geqq 1)$, if

$$\sum n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty \quad ([1]). \tag{1.1}$$

Summability $|C, \alpha|_1$ is the same as summability $|C, \alpha|$.
Since

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha),$$

condition (1.1) can also be written as

$$\sum \frac{|t_n^\alpha|^k}{n} < \infty. \tag{1.2}$$

If

$$\sum_1^n \frac{|s_\nu|}{\nu} = O(\log n), \tag{1.3}$$

as $n \to \infty$, then Σa_n is said to be strongly bounded by logarithmic means with index 1, or bounded $[R, \log n, 1]$.

2. Recently PATI [2] proved the following theorem concerning summability $|C, 1|$ of a factored infinite series.

Let $\{\lambda_n\}$ be a convex sequence such that $\Sigma \frac{\lambda_n}{n}$ is convergent (then, necessarily, $\lambda_n \geqq 0$). *If Σa_n is bounded $[R, \log n, 1]$, then $\Sigma a_n \lambda_n$ is summable $|C, 1|$.*

The object of this note is to generalize this result by obtaining a theorem for summability $|C, 1|_k$.

3. In what follows we shall establish the following theorem.

Theorem. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \frac{\lambda_n}{n} < \infty$, and*

$$\sum_1^n |s_\nu|^k/\nu = O(\log n) \qquad (k \geqq 1), \tag{3.1}$$

then $\Sigma a_n \lambda_n$ is summable $|C, 1|_k$.

It is clear that in the special case $k=1$ our theorem includes the above theorem of PATI. For $k>1$ $\left(\frac{1}{k}+\frac{1}{k'}=1\right)$, we observe that

$$\sum_1^n \frac{|s_\nu|}{\nu} \leqq \left(\sum_1^n \frac{|s_\nu|^k}{\nu}\right)^{1/k} \left(\sum_1^n \frac{1}{\nu}\right)^{1/k'} = O\{(\log n)^{1/k}(\log n)^{1/k'}\} = O(\log n).$$

Thus condition (3. 1) implies condition (1. 3). However the results of FLETT [1] show that summability $|C, 1|_k$ and summability $|C, 1|$ in general are independent of each other.

4. The following lemmas will be required for the proof of this theorem.

Lemma 1. [2] *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum \frac{\lambda_n}{n}<\infty$, *then*

$$\Sigma \log (n+1)\Delta\lambda_n < \infty$$

and

$$m \log (m+1)\Delta\lambda_m = O(1),$$

as $m\to\infty$.

Lemma 2. [2] *Under the condition of Lemma* 1, *we have*

$$\sum_1^m n\log (n+1)\Delta^2 \lambda_n = O(1), \quad as \quad m\to\infty.$$

5. Proof of the Theorem. Let T_n denote the n-th Cesàro mean of order 1 of the sequence $\{na_n\lambda_n\}$. Then we have to show that

$$\sum_1^\infty n^{-1}|T_n|^k<\infty. \tag{5. 1}$$

Now,

$$T_n = \frac{1}{n+1}\sum_{\nu=1}^n \nu a_\nu\lambda_\nu = \frac{1}{n+1}\sum_1^{n-1} \Delta(\nu\lambda_\nu)s_\nu + \frac{ns_n\lambda_n}{n+1} - \frac{a_0\lambda_1}{n+1} =$$

$$= \frac{1}{n+1}\sum_{\nu=1}^n \Delta(\nu\lambda_\nu)s_\nu - \frac{s_n}{n+1}\left(n\lambda_n-(n+1)\lambda_{n+1}\right) + \frac{ns_n\lambda_n}{n+1} - \frac{a_0\lambda_1}{n+1} =$$

$$= \frac{1}{n+1}\sum_1^n \Delta(\nu\lambda_\nu)s_\nu + s_n\lambda_{n+1} - \frac{a_0\lambda_1}{n+1} = L_1^{(n)}+L_2^{(n)}+L_3^{(n)}.$$

By MINKOWSKI's inequality it is therefore sufficient to prove that

$$\sum \frac{|L_1^{(n)}|^k}{n}<\infty, \tag{5. 2}$$

$$\sum \frac{|L_2^{(n)}|^k}{n}<\infty, \tag{5. 3}$$

$$\sum \frac{|L_3^{(n)}|^k}{n}<\infty. \tag{5. 4}$$

Proof of (5. 2). In the sequel C_1, C_2 denote positive constants. We have

$$\sum_1^\infty \frac{|L_1^{(n)}|^k}{n} = \sum_1^\infty \frac{1}{n(n+1)^k}\left|\sum_1^n (\Delta\nu\lambda_\nu)s_\nu\right|^k \leqq \sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n |\Delta\nu\lambda_\nu|\,|s_\nu|\right)^k \leqq$$

$$\leqq C_1\sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n \nu\Delta\lambda_\nu|s_\nu|\right)^k + C_1\sum_1^\infty \frac{1}{n^{k+1}}\left(\sum_1^n \lambda_{\nu+1}|s_\nu|\right)^k \leqq$$

$$\leqq C_1\sum_1^\infty \frac{1}{n^{k+1}}\sum_1^n \nu\Delta\lambda_\nu|s_\nu|^k\left(\sum_1^n \nu\Delta\lambda_\nu\right)^{k/k'} + C_1\sum_1^\infty \frac{1}{n^{k+1}}\sum_{\nu=1}^n \lambda_{\nu+1}|s_\nu|^k\left(\sum_{\nu=1}^n \lambda_{\nu+1}\right)^{k/k'} =$$

$$= O\left(\sum_1^\infty \frac{1}{n^2}\sum_{\nu=1}^n \nu\Delta\lambda_\nu|s_\nu|^k\right) + O\left(\sum_{n=1}^\infty \frac{1}{n^2}\sum_{\nu=1}^n \lambda_\nu|s_\nu|^k\left(\sum_1^\infty \frac{\lambda_\nu}{\nu}\right)^{k/k'}\right) =$$

$$= O\left(\sum_{\nu=1}^\infty \nu\Delta\lambda_\nu|s_\nu|^k\sum_{n=\nu}^\infty \frac{1}{n^2}\right) + O\left(\sum_{\nu=1}^\infty \lambda_\nu|s_\nu|^k\sum_{n=\nu}^\infty \frac{1}{n^2}\right) =$$

$$= O\left(\sum_{\nu=1}^\infty \Delta\lambda_\nu|s_\nu|^k\right) + O\left(\sum_{\nu=1}^\infty \frac{\lambda_\nu}{\nu}|s_\nu|^k\right).$$

Now

$$\sum_{\nu=1}^m \Delta\lambda_\nu|s_\nu|^k = \sum_1^m \nu\Delta\lambda_\nu\frac{|s_\nu|^k}{\nu} =$$

$$= \sum_1^{m-1} \Delta(\nu\Delta\lambda_\nu)\sum_{\mu=1}^\nu \frac{|s_\mu|^k}{\mu} + m\Delta\lambda_m\sum_{\mu=1}^m \frac{|s_\mu|^k}{\mu} =$$

$$= O\left(\sum_1^{m-1} \nu\Delta^2\lambda_\nu\log(\nu+1)\right) + O\left(\sum_1^{m-1} \Delta\lambda_\nu\log(\nu+1)\right) + O(m\Delta\lambda_m\log(m+1)) = O(1),$$

by virtue of Lemmas 1 and 2.

Also applying Lemma 1 we have

$$\sum_1^m \frac{\lambda_\nu}{\nu}|s_\nu|^k = \sum_1^{m-1} \Delta\lambda_\nu\sum_{\mu=1}^\nu \frac{|s_\mu|^k}{\mu} + \lambda_m\sum_{\mu=1}^m \frac{|s_\mu|^k}{\mu} =$$

$$= O\left(\sum_1^{m-1} \Delta\lambda_\nu\log(\nu+1)\right) + O(\lambda_m\log(m+1)) = O(1).$$

Hence

$$\sum_1^\infty \frac{|L_1^{(n)}|^k}{n} = O(1).$$

Proof of (5. 3). We have by virtue of the hypothesis

$$\sum_1^m \frac{|L_2^{(n)}|^k}{n} = \sum_1^m \frac{1}{n}|s_n\lambda_{n+1}|^k \leqq \sum_1^m \lambda_n^k\frac{|s_n|^k}{n} = \sum_1^{m-1} \Delta\lambda_n^k\sum_{\mu=1}^n \frac{|s_\mu|^k}{\mu} + \lambda_m^k\sum_1^m \frac{|s_\mu|^k}{\mu} =$$

$$= \sum_1^{m-1} \Delta\lambda_n^k O(\log n) + O(\lambda_m^k\log m) = O\left(\sum_1^m \Delta\lambda_n^k\log n\right) + O(1) = O(1).$$

Finally, it is clear that

$$\sum \frac{|L_3^{(n)}|^k}{n} \leqq C_2 \sum_1^{\infty} \frac{1}{n^{k+1}} < \infty.$$

This completes the proof of the theorem.

The author would like to express his warmest thanks to Professor B. N. PRASAD for his kind encouragement and helpful suggestions.

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On the strong summability of orthogonal series

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Let $\{\varphi_\nu(x)\}$ $(\nu=0, 1, 2, \ldots)$ be a normalized orthogonal system in $[a, b]$ and

$$\sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x) \tag{1}$$

be an orthogonal series which satisfies

$$\sum_{\nu=0}^{\infty} c_\nu^2 < \infty. \tag{2}$$

We write

$$s_n(x) = \sum_{\nu=0}^{n} c_\nu \varphi_\nu(x)$$

and

$$\sigma_n^\alpha(x) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu(x) \qquad \left(A_n^\alpha = \binom{n+\alpha}{n}\right).$$

Concerning the strong summability of orthogonal series, TANDORI proved the following theorems.

Theorem A. (TANDORI [3]) *If the orthogonal series* (1) *with* (2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* $[a, b]$, *then*

$$\lim_{n\to\infty} \sigma_{2^n}^\alpha([s_\nu - f]^2; x) = 0 \qquad (0<\alpha<1)$$

almost everywhere in $[a, b]$, *where*

$$\sigma_n^\alpha([s_\nu - f]^2; x) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} [s_\nu(x) - f(x)]^2.$$

Theorem B. (TANDORI [4]) *If*

$$\sum_{\nu=2}^{\infty} c_\nu^2 (\log\log \nu)^2 < \infty$$

then there exists a square-integrable function $f(x)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} [s_{n_\nu}(x) - f(x)]^2 = 0$$

almost everywhere for any increasing sequence n_ν.

In the present note, the author intends to generalize these results to the following form.

Theorem 1. *If the orthogonal series* (1) *with* (2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* $[a, b]$, *then*

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x) - f(x)|^k = 0$$

almost everywhere in $[a, b]$ *for any* $\alpha > 0$ *and* $k > 0$.

Theorem 2. *If*

$$\sum_{\nu=2}^{\infty} c_\nu^2 (\log \log \nu)^2 < \infty,$$

then there exists a square-integrable function $f(x)$ *such that*

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{n_\nu}(x) - f(x)|^k = 0$$

for any $\alpha > 0$ *and* $k > 0$, *almost everywhere in* $[a, b]$ *for any increasing sequence* n_ν.

In the sequel, we use $A, B, \ldots$ to denote positive constants, not necessarily the same on any two occurences.

Lemma 1. *If* $\sum_{\nu=0}^{\infty} c_\nu^2 < \infty$, *then*

$$\int_a^b \left\{ \sum_{n=1}^{\infty} n^{-1} |\sigma_n^{\alpha-1}(x) - \sigma_n^\alpha(x)|^2 \right\} dx \leqq A \sum_{n=0}^{\infty} c_n^2 \qquad (\alpha > 1/2).$$

Proof. This is well known. In fact, it can be proved by application of BESSEL's inequality.

For any scalar series

$$\sum_{\nu=0}^{\infty} a_\nu, \tag{3}$$

we write

$$s_n = \sigma_n^0 = \sum_{\nu=0}^{n} a_\nu,$$

$$s_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu, \quad \sigma_n^\alpha = s_n^\alpha / A_n^\alpha, \quad \sigma_n = \sigma_n^1.$$

Then we have the identity

$$n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) = -\alpha(\sigma_n^\alpha - \sigma_n^{\alpha-1}) \qquad (\alpha > 0)$$

and we write this τ_n^α. If

$$\sum_{n=1}^{\infty} n^{-1}|\tau_n^\alpha|^k<\infty,$$

then FLETT [1] says that the series (3) is summable $|C, \alpha|_k$.

Lemma 2. *If $\Sigma c_n^2<\infty$, then*

$$\int_a^b \{\sum_{n=1}^{\infty} n^{-1}|s_n(x)-\sigma_n(x)|^r\}^{2/r}\,dx \leqq B\sum_{n=0}^{\infty} c_n^2 \qquad (r\geqq 2),$$

and (1) *is $|C, 1|_r$-summable almost everywhere for any $r\geqq 2$.*

Proof. FLETT [1, p. 115] proved that

$$\left(\sum_{n=1}^{\infty} n^{-1}|\tau_n^\beta|^r\right)^{1/r} \leqq A(k, r, \alpha, \beta)\left(\sum_{n=1}^{\infty} n^{-1}|\tau_n^\alpha|^k\right)^{1/k}$$

whenever

$$r\geqq k>1,\ \alpha>-1,\ \beta\geqq\alpha+1/k-1/r.$$

In particular, setting $k=2$, $\alpha=1/2+\varepsilon$ $(\varepsilon>0)$, we have

$$\beta\geqq 1+\varepsilon-1/r.$$

For any given $r\geqq 2$, we can select $\varepsilon<r^{-1}$. Hence we have

$$\left(\sum_{n=1}^{\infty} n^{-1}|\tau_n^1|^r\right)^{1/r} \leqq A\left(\sum_{n=1}^{\infty} n^{-1}|\tau_n^\alpha|^2\right)^{1/2} \qquad (\alpha=1/2+\varepsilon).$$

By Lemma 1, for $\alpha>1/2$, $r\geqq 2$,

$$\int_a^b\left\{\sum_{n=1}^{\infty} n^{-1}|s_n(x)-\sigma_n(x)|^r\right\}^{2/r}dx \leqq A\int_a^b\left\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)|^2\right\}dx \leqq B\sum_{n=0}^{\infty} c_n^2.$$

Hence

$$\sum_{n=1}^{\infty} n^{-1}|s_n(x)-\sigma_n(x)|^r$$

converges almost everywhere.

Lemma 3. *If $\Sigma c_n^2<\infty$, then*

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k\right)^{1/k}\right\}^2 dx \leqq A\sum_{n=0}^{\infty} c_n^2.$$

Proof. For any given $\alpha>0$, we select s near to 1 such as $\alpha>1-s^{-1}(s>1)$ and set $r^{-1}+s^{-1}=1$. Then, by HÖLDER's inequality,

$$\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k \leqq \frac{1}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}\left\{\sum_{\nu=1}^{n}\nu^{s/r}(A_{n-\nu}^{\alpha-1})^s\right\}^{1/s} \leqq$$

$$\leqq \frac{B}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}\{n^{s/r}n^{(\alpha-1)s+1}\}^{1/s} \leqq$$

$$\leqq \frac{Bn^\alpha}{A_n^\alpha}\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r} \leqq C\left\{\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right\}^{1/r}.$$

Hence
$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k\right)^{1/k}\right\}^2 dx \leqq$$

$$\leqq C\int_a^b\left(\sum_{\nu=1}^{\infty}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{2/rk}dx.$$

For a given k, we take s sufficiently near to 1, then rk is greater than 2 because $r^{-1}+s^{-1}=1$. Hence, by Lemma 2, we get the required result.

The method of proof of Lemma 3 has been given in SUNOUCHI and YANO [2].

Proof of Theorem 1. For any positive $\varepsilon>0$, we take

$$\sum_{\nu=N+1}^{\infty}c_\nu^2<\varepsilon^3$$

and split Σc_ν^2 into

$$\sum_{\nu=1}^{N}c_\nu^2 \quad\text{and}\quad \sum_{\nu=N+1}^{\infty}c_\nu^2.$$

Consider the orthogonal series

$$\sum_{n=1}^{N}c_n\varphi_n(x) \tag{4}$$

and

$$\sum_{n=N+1}^{\infty}c_n\varphi_n(x). \tag{5}$$

For the series (4), the conclusion is valid evidently and for the second series (5),

$$\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k\right)^{1/k}\right\}^2dx \leqq A\sum_{\nu=N+1}^{\infty}c_\nu^2<A\varepsilon^3,$$

where $t_n(x)$ and $T_n(x)$ is the partial sums and the arithmetic means of (5). Hence

$$\operatorname{meas}\left\{x \mid \limsup\left(\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k\right)^{1/k} > \varepsilon\right\} \leqq A\varepsilon.$$

That is,
$$\lim_{n\to\infty}\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}|t_\nu(x)-T_\nu(x)|^k = 0$$

almost everywhere for the series (5). Combining the two results for series (4) and (5) we conclude that if
$$\Sigma c_n^2 < \infty,$$
then
$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-\sigma_\nu(x)|^k \qquad (\alpha>0,\ k>0)$$

converges to zero almost everywhere. By the hypothesis, $\sigma_n(x)$ converges to $f(x)$ a.e. and so
$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_\nu(x)-f(x)|^k \qquad (\alpha>0,\ k>0)$$

converges to zero almost everywhere.

Proof of Theorem 2. Let us set
$$\psi_\nu(x) = \frac{1}{\gamma_n}[c_{n_\nu+1}\varphi_{n_\nu+1}(x)+\dots+c_{n_{\nu+1}}\varphi_{n_{\nu+1}}(x)]$$
where
$$\gamma_\nu = (c_{n_\nu+1}^2+\dots+c_{n_{\nu+1}}^2)^{1/2}.$$

Then $\{\psi_\nu(x)\}$ $(\nu=0, 1, 2, \dots)$ also is a normalized orthogonal system and the orthogonal series

(6) $$\sum_{\nu=0}^{\infty}\gamma_\nu\psi_\nu(x)$$

satisfies
$$\Sigma\gamma_\nu^2 < \infty.$$

Since n_ν is an increasing sequence and $n_\nu \geqq \nu$,

$$\sum_{\nu=2}^{\infty}\gamma_\nu^2(\log\log\nu)^2 = \sum_{\nu=2}^{\infty}(c_{n_\nu+1}^2+\dots+c_{n_{\nu+1}}^2)(\log\log\nu)^2 \leqq$$

$$\leqq \sum_{\nu=2}^{\infty}(c_{n_\nu+1}^2(\log\log(n_\nu+1))^2+\dots+c_{n_{\nu+1}}^2(\log\log n_{\nu+1})^2) \leqq \sum_{n=2}^{\infty}c_n^2(\log\log n)^2 < \infty,$$

by the hypothesis.

By the well-known theorem, the sequence of the $(C, 1)$-means of the series (6) tends to $f(x)$ almost everywhere. Moreover,
$$\sum_{\nu=0}^{m}\gamma_\nu\psi_\nu(x) = \sum_{n=0}^{n_m}c_n\varphi_n(x)$$

and so the m-th ordinary partial sum of (6) is identical with the n_m-th partial sum of (1). Applying Theorem 1, we get the required.

Remark. Actually our argument proves the following maximal theorem:

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{n_\nu}(x) - f(x)|^k \right)^{1/k} \right\}^2 dx \leqq A \sum_{n=2}^{\infty} c_n^2 (\log \log n)^2,$$

for any $\alpha > 0$ and $k > 0$.

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Über das Rieszsche Eindeutigkeitskriterium des Momentenproblems

Von G. FREUD in Budapest

Es handelt sich um den folgenden, von M. RIESZ[1]) herrührenden Satz:

Unter der Bedingung

$$\liminf_{n\to\infty}\left(\sqrt[2n]{\mu_{2n}}/2n\right)<\infty \tag{1}$$

hat das Momentenproblem

$$\int_{-\infty}^{+\infty} x^n\,d\alpha(x)=\mu_n \qquad (n=0,1,\dots)\,^{2)} \tag{2}$$

höchstens eine einzige, nicht abnehmende Lösungsfunktion $\alpha(x)$ mit den Normierungseigenschaften

$$\alpha(-\infty)=0,\quad \alpha(x)=\frac{\alpha(x+0)+\alpha(x-0)}{2}. \tag{3}$$

Wir wollen hierfür einen neuen Beweis angeben.

Infolge (1) gibt es eine gegen ∞ strebende Folge von geraden Zahlen N_ν und eine positive Zahl R mit

$$\mu_{N_\nu}<R^{N_\nu}N_\nu^{N_\nu} \qquad (\nu=1,2,\dots). \tag{4}$$

Wegen der Stirlingschen Formel

$$n!\sim n^n e^{-n}\sqrt{2\pi n}$$

gilt andererseits

$$n!>n^n e^{-2n} \quad \text{für} \quad n\geqq n_0. \tag{5}$$

Durch Unterdrücken endlich vieler Glieder der Folge $\{N_\nu\}$ (falls nötig) können wir annehmen, daß $N_\nu\geqq n_0$ für jedes ν.

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[1]) M. RIESZ, Sur le problème des moments, Troisième Note, *Arkiv för Mat. Astr. och Fysik*, **17**, No. 16 (1923); vgl. S. 48.

Der Satz von M. RIESZ ist in einem allgemeineren Satze von TH. CARLEMAN enthalten, angekündigt in *Comptes Rendus Acad. Sci. Paris*, **174** (1922), 1680—1682, und bewiesen im Buch: TH. CARLEMAN, *Les fonctions quasi-analytiques* (Paris, 1926), S. 81.

[2]) $\mu_0, \mu_1, \dots$ ist eine vorgegebene Folge von reellen Zahlen und man setzt voraus, daß die Integrale absolut konvergent sind. Wegen (2) ist notwendigerweise $\mu_{2n}\geqq 0$ für $n=0,1,\dots$.

Auf Grund der Taylorschen Formel mit Lagrangeschem Restglied gilt für jede reelle Zahl x und für jede natürliche Zahl N:

$$e^{ix} = \cos x + i \sin x = \sum_{n=0}^{N-1} \frac{(ix)^n}{n!} + \frac{x^N}{N!} [\cos^{(N)} \theta_1 x + \sin^{(N)} \theta_2 x] \qquad (0 < \theta_1, \theta_2 < 1),$$

also ist

$$\left| e^{ix} - \sum_{n=0}^{N-1} \frac{i^n x^n}{n!} \right| \leqq \sqrt{2} \frac{|x|^N}{N!}.$$

Sind x, t und t_0 reell, so folgt hieraus

$$\left| e^{i(t-t_0)x} - \sum_{n=0}^{N-1} \frac{[i(t-t_0)x]^n}{n!} \right| \leqq \sqrt{2} \frac{|(t-t_0)x|^N}{N!}$$

und — durch Multiplizieren mit $e^{it_0 x}$ —

$$\left| e^{itx} - \sum_{n=0}^{N-1} \frac{i^n (t-t_0)^n x^n}{n!} e^{it_0 x} \right| \leqq \sqrt{2} \frac{|t-t_0|^N |x|^N}{N!}. \tag{7}$$

Wir nehmen nun an, $\alpha(x)$ sei eine nichtabnehmende, den Nebenbedingungen (3) genügende Lösung des Momentenproblems (2) und wir setzen

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} \, d\alpha(x) \qquad (-\infty < t < \infty).$$

Da die Integrale (2) absolut konvergent vorausgesetzt sind, ist $\varphi(t)$ unendlich oft differenzierbar und man hat

$$\varphi^{(n)}(t) = i^n \int_{-\infty}^{+\infty} x^n e^{itx} \, d\alpha(x) \qquad (n = 0, 1, \ldots). \tag{8}$$

Auf Grund von (8) und der Ungleichung (7) bekommt man durch Integration

$$\left| \varphi^{(n)}(t) - \sum_{n=0}^{N-1} \frac{(t-t_0)^n}{n!} \varphi^{(n)}(t_0) \right| \leqq \sqrt{2} \frac{|t-t_0|^N}{N!} \int_{-\infty}^{+\infty} |x|^N \, d\alpha(x). \tag{9}$$

Für gerade N ist hier die rechte Seite gleich $\sqrt{2}|t-t_0|^N \mu_N / N!$. Wegen (4) und (5) ist also für $N = N_\nu$ die rechte Seite kleiner als

$$\sqrt{2}(t-t_0)^N \frac{R^N N^N}{N^N e^{-2N}} = \sqrt{2} \left(\frac{t-t_0}{r} \right)^N \tag{10}$$

mit $r = (e^2 R)^{-1}$.

Wenn $N = N_\nu$, $\nu \to \infty$ und $|t-t_0| < r$, so strebt die Größe (10) gegen 0. Also gilt

$$\varphi(t) = \lim_{\nu \to \infty} \sum_{n=0}^{N_\nu - 1} \frac{(t-t_0)^n}{n!} \varphi^{(n)}(t_0) \quad \text{für} \quad |t-t_0| < r. \tag{11}$$

Wegen (8) hat man insbesondere $\varphi^{(n)}(0)=i^n\mu_n$, also nimmt (11) für $t_0=0$ die folgende Gestalt an:

$$\varphi(t) = \lim_{\nu\to\infty} \sum_{n=0}^{N'_\nu-1} \frac{(it)^n}{n!}\mu_n.$$

Diese Formel zeigt, daß die Funktion $\varphi(t)$ auf dem Intervall $(-r, +r)$ durch die Momente μ_n eindendig bestimmt wird. Wenn aber die Funktion $\varphi(t)$ auf irgendeinem Intervall (a, b) bestimmt ist, so folgt aus (11), daß $\varphi(t)$ auch im Intervall $(a-r, b+r)$ bestimmt ist; man lasse nun t_0 das Intervall (a, b) durchlaufen. Man schließt, daß $\varphi(t)$ durch die Momente μ_n auf der ganzen Geraden $(-\infty, +\infty)$ eindeutig bestimmt ist.

Durch die bekannte Umkehrformel

$$\alpha(x)-\alpha(y) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{+T} \varphi(t) \frac{e^{-ity}-e^{-itx}}{it} dt$$

(wobei x, y beliebige Stetigkeitsstellen von $\alpha(x)$ sind) und wegen den Normierungsbedingungen (3) wird $\alpha(x)$ durch $\varphi(t)$, also durch die Momente μ_n eindeutig bestimmt.

Damit ist der Beweis fertig.

Herrn Prof. BÉLA SZ.-NAGY bin ich für wertvolle Ratschläge zu Dank verpflichtet.

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On the characterization of Stone lattices

By JULES VARLET in Liège (Belgium)

1. Introduction

M. H. STONE proposed the following problem:

"What is the most general pseudo-complemented distributive lattice in which $a^* \vee a^{**} = 1$ identically?". See [1], p. 149, problem 70.

The first solution to this problem was given by G. GRÄTZER and E. T. SCHMIDT in [3]:

"Let L be a distributive pseudo-complemented lattice with unit element. Then L is a Stone lattice if and only if the lattice-theoretical join of any two distinct minimal prime ideals of L is L."

In this note, we give a short proof of an equivalent form of this theorem and of the corresponding one for relative Stone lattices.*)

The reader is referred to [3] and [4] for the notions and notations. We only replace the words "dual ideal" by "filter".

2. Stone lattices

Theorem I. *A distributive pseudo-complemented lattice is a Stone lattice if and only if every prime filter is contained in only one proper maximal filter.*

Proof. 1°) *if*: Let us suppose that L is a distributive pseudo-complemented lattice which satisfies the above condition but which is not a Stone lattice. Then there exists an element a such that $a^* \vee a^{**} = b < 1$. By STONE's theorem (cf. [3], lemma I), there exists a prime filter P, containing 1 and disjoint from the principal ideal $(b]$.

Let us consider the filter $F = P \vee [a^*)$. F cannot contain a^{**}; otherwise $0 = a^* \wedge a^{**}$ would belong to F and there would exist an element $x \in P$ such that $x \wedge a^* = 0$. But this last equality implies $x \leqq a^{**}$, hence $x \leqq b$, contradicting that P is disjoint from $(b]$.

The family of filters containing F and disjoint from $(a^{**}]$ has a maximal element F_1 and this filter F_1 is a maximal proper filter of L, for any filter containing properly F_1 would contain a^{**}, and consequently 0 (because it also contains a^*).

*) The author wishes to express his indebtedness to Professor G. GRÄTZER for his encouragement and advice in the preparation of this paper.

In the same way, we can show that there exists a maximal proper filter G_1 containing the filter $G = P \vee [a^{**})$ but not containing a^*.

Thus the prime filter P would be contained in two distinct maximal filters, contradicting the hypothesis.

2°) *only if*: Let us suppose that there exists in the Stone lattice L a prime filter P contained in two distinct maximal proper filters M_1 and M_2. Let a be an element belonging to M_1 but not to M_2. Since $a \wedge a^* = 0$, therefore $a^* \notin M_1$ and $a^* \notin P$. Since M_2 is maximal, for any element not belonging to it, there exists in M_2 an element which is disjoint from the first. Thus there exists $b \in M_2$ such that $b \wedge a = 0$. Since $a^* \geqq b$, we have: $a^* \in M_2$ and $a^{**} \notin M_2$, $a^{**} \notin P$. Then $a^* \vee a^{**} = 1 \in P$, and $a^*, a^{**} \notin P$, contradicting that P is a prime filter.

3. Equivalent propositions

By dualizing the statements, it is easy to show that the condition

(A) *any prime filter is contained in only one maximal filter*

is equivalent, in a distributive lattice with elements 0 and 1, to the condition

(A′) *any prime ideal contains only one minimal prime ideal.*

Then we can verify the equivalence, in any distributive lattice, of the condition (A′) to the condition cited above:

(B) *the lattice-theoretical join of any two distinct minimal prime ideals of L is L.*

Proof. (A′)⇒ (B). Let us consider, in the distributive lattice L, the two minimal prime ideals P and Q such that $P \vee Q \neq L$. There exists a prime filter F disjoint from $P \vee Q$. But then $L - F$ is a prime ideal containing P and Q, contradicting (A′).

(B) ⇒ (A′). Let us suppose that, in the distributive lattice L, the prime ideal P contains two minimal prime ideals Q and R. We would have $Q \vee R \subseteqq P$ and (B) would be contradicted.

4. Relative Stone lattices

Theorem 2. *A distributive lattice in which every closed interval (as a sublattice) is a pseudo-complemented lattice, is a relative Stone lattice if and only if every proper filter which contains a prime filter is prime.*

Proof. 1°) *if*: Let us suppose that L is a distributive lattice satisfying the conditions of the hypothesis which is not a relative Stone lattice. There exists in L an interval $[k, l]$ in which a prime filter F' is contained in two distinct maximal filters G' and H'. Consider the mapping $x \to x' = (x \vee k) \wedge l$ of L onto $[k, l]$. Since L is distributive, this mapping is an endomorphism. Let F, G and H be the inverse images of F', G' and H' respectively. Obviously F, G and H are filters. Moreover, $G \supset F$, $H \supset F$, G and H are non-comparable. By the lemma III of [4], F is a prime filter. Thus we come to a contradiction since the prime filter F is contained in a proper filter $G \wedge H$, which is not prime*).

*) In a distributive lattice, a filter is prime if and only if it is $\wedge$-irreducible.

2) *only if:* Let us suppose that in the relative Stone lattice L there exists a prime filter F contained in a non-prime proper filter G. G being non-prime, there would be in $L-G$ two elements a and b such that $a \vee b = d \in G$. More precisely, d belongs to $G-F$ since F is prime. Let $e \in F$, $e > d$. Let us put $a \wedge b = c$. By hypothesis, the interval $[c, e]$ is a Stone lattice. We have: $a^* \vee a^{**} = e$ and $b \leqq a^*$ (where * denotes the pseudo-complement in $[c, e]$). Since $d \wedge a^* = (a \vee b) \wedge a^* = b \wedge a^* = b$ and $d \in G$, $b \notin G$, we conclude: $a^* \notin G$ (and $a^* \notin F$). A similar argument shows that $a^{**} \notin F$. Since F is a prime filter, this is a contradiction.

5. Equivalent propositions

The condition

(C) *any proper filter which contains a prime filter is prime*

is equivalent, in a distributive lattice, to the condition of G. GRÄTZER and E. T. SCHMIDT (cf. [3], theorem 3):

(D) *for any pair of prime ideals P and Q, neither of which contains the other, $P \vee Q$ is the whole lattice.*

Proof. (C) ⇒ (D). Let us suppose that, in the distributive lattice L, there exist two non-comparable prime ideals A and B such that $A \vee B \neq L$. By Stone's theorem, there exists a prime filter F disjoint from the ideal $A \vee B$. $L-A$ and $L-B$ are non-comparable prime filters the intersection of which is a filter G containing F. By assumption (C), this filter G is prime, which is impossible since that would imply that G is $\wedge$-irreducible.

(D) ⇒ (C). Again, let us demonstrate this implication in an indirect way. Let us assume the existence, in the distributive lattice L, of a prime filter F and a non-prime proper filter G containing F. Thus G is $\wedge$-reducible: there exist non-comparable filters A and B such that $A \wedge B = G$. Thus we can find two elements a and b such that $a \in A$, $a \notin B$, $b \in B$, $b \notin A$. There exists a prime filter A_1 containing A and disjoint from $(b]$ and a prime filter B_1 containing B and disjoint from $(a]$. A_1 and B_1 contain G and are non-comparable.

The non-comparable prime ideals $L-A_1$ and $L-B_1$ should have, by (D), the lattice-theoretical join L. This would imply that any element of F is the join of two elements not belonging to F, hence a contradiction.

In conclusion, let us recall that A. MONTEIRO gives in [6] two conditions, equivalent to (C) in a distributive lattice:

(C′) *the family of all filters including a prime filter is linearly ordered;*

(C″) *the family of all prime filters including a prime filter is linearly ordered.*

The equivalence of conditions (C), (C′) and (C″) can be easily proved.

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Proximity structures in Boolean Algebras

By K. N. MEENAKSHI in Madras (India)

This paper is an attempt to study the notion of proximity structure (cf. I. S. GÁL) in the class of classical topological Boolean Algebras (cf. G. NÖBELING). A topological proximity Boolean algebra is defined in a manner similar to that of a proximity space. In the first section we prove that a classical topological Boolean algebra is completely regular if and only if it is a topological proximity algebra. Then we proceed to show that there exists a coarsest uniform structure compatible with a proximity structure of a classical topological Boolean algebra. Using this we prove that a classical topological Boolean algebra is completely regular if and only if it is homeomorphic to an invariant subalgebra of a compact regular space.

In section 2 we study quotient algebras of the form $S(X)/I$ where X is a completely regular space of topological weight m and I is an m-additive ideal of $S(X)$ (cf. SIKORSKI). We introduce the notion of quotient uniformity and quotient proximity in $S(X)/I$ and discuss the permutability of the two operations of taking quotient proximity and quotient uniformity.

A similar concept is studied by A. S. ŠVARC (cf. *Math. Reviews*, **19** (1958), p. 436) in his paper on "Proximity spaces and lattices". In Section 3 the connection between the concept of δ-lattices of ŠVARC and our notion of proximity Boolean algebras is explained.

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1. Definition 1. A *proximity relation* $\bar{\delta}$ for a Boolean algebra $\mathfrak{B}$ is a binary relation which satisfies the following axioms:

P. 1: $A\bar{\delta}0$ for every element $A\in\mathfrak{B}$ where 0 is the zero element of $\mathfrak{B}$.

P. 2: $A_1\bar{\delta}A_2 \Leftrightarrow A_2\bar{\delta}A_1$ for any two elements A_1, A_2 in $\mathfrak{B}$.

P. 3: $A_1\wedge A_2>0 \Rightarrow A_1\delta A_2$ (i.e. not $A_1\bar{\delta}A_2$), where $A_1\wedge A_2$ denotes the Boolean product of A_1, A_2 in $\mathfrak{B}$.

P. 4: $A\bar{\delta}(B+C) \Leftrightarrow A\bar{\delta}B$ and $A\bar{\delta}C$ where $B+C$ denotes the Boolean sum of B and C in $\mathfrak{B}$.

P. 5: $A_1\bar{\delta}A_2 \Rightarrow$ there exist elements B_1, B_2 in $\mathfrak{B}$ such that $A_i\bar{\delta}cB_i$ for $i=1, 2$ and $B_1\bar{\delta}B_2$ where cB denotes the complement of B in $\mathfrak{B}$.

Definition 2. Let $(\mathfrak{B}, \tau)$ be a classical topological Boolean algebra. Then a proximity relation $\bar{\delta}$ defined in $\mathfrak{B}$ is said to be *compatible with the topology* τ on $\mathfrak{B}$ if for each element A in $\mathfrak{B}$, $\Sigma(U|U\bar{\delta}cA)$ exists in $\mathfrak{B}$ and $\text{int}A=\Sigma(U|U\bar{\delta}cA)$.

Note. The proximity relation $\bar{\delta}$ is compatible with the topology τ of $(\mathfrak{B}, \tau)$ if and only if the open elements of $(\mathfrak{B}, \tau)$ are precisely the elements (A) of the form $A = \Sigma(U | U\bar{\delta}cA)$.

$(\mathfrak{B}, \tau)$ is called a topological proximity algebra if there exists a proximity relation on $\mathfrak{B}$ compatible with its topology τ.

Definition 3. (See NÖBELING, p. 91.) Let $(\mathfrak{B}, \tau)$ be a classical topological Boolean algebra. If for each dyadic rational $t = m/2^n$ ($m = 0, 1, 2, \dots, 2^n$; $n = 1, 2, \dots$) H_t is an open element of $\mathfrak{B}$ such that $\bar{H}_{t'} < H_{t''}$ for $t' < t''$, then we call the set $\{H_t\}$ of open elements H_t from $\mathfrak{B}$ a *binary scale.*

Definition 4. A classical topological Boolean algebra $\mathfrak{B}$ is *completely regular* if for any two non-zero elements A_0 and F_1 of $\mathfrak{B}$ such that (1) $A_0 \wedge F_1 = 0$ and (2) F_1 is closed, there exists a binary scale (H_t) such that $A_0 \wedge H_0 > 0$ and $F_1 \wedge H_1 = 0$.

Proposition 1. *Let $(\mathfrak{B}, \tau, \bar{\delta})$ be a classical topological proximity Boolean algebra. Then*

(1) $A_i \leqq B_i$ $(i = 1, 2)$ *and* $B_1\bar{\delta}B_2 \Rightarrow A_1\bar{\delta}A_2$;

(2) $A_1\bar{\delta}A_2 \Rightarrow \bar{A}_1\bar{\delta}\bar{A}_2$ *where $\bar{A}$ denotes the closure of A in $(\mathfrak{B}, \tau)$;*

(3) $A_1\bar{\delta}cA_3 \Rightarrow$ *there exists an element A_2 in $\mathfrak{B}$ such that $A_1\bar{\delta}cA_2$ and $A_2\bar{\delta}cA_3$;*

(4) $A_1\bar{\delta}A_2 \Rightarrow$ *there exist open elements G_1, G_2 in $\mathfrak{B}$ with $A_i\bar{\delta}cG_i$, $i = 1, 2$ and $G_1\bar{\delta}G_2$; and*

(5) $(\mathfrak{B}, \tau)$ *is completely regular.*

Proof. (1), (2), (3) and (4) follow by simple arguments using P_1, P_2, P_3, P_4 and P_5. For example we shall prove (2). To prove (2) it suffices to show that $A_1\bar{\delta}A_2 \Rightarrow \bar{A}_1\bar{\delta}A_2$. $A_1\bar{\delta}A_2 \Rightarrow$ there exist elements B_1, B_2 with $A_i\bar{\delta}cB_i$, $i = 1, 2$, and $B_1\bar{\delta}B_2 \Rightarrow$ there exist open elements H_1, H_2 such that $A_i < H_i$ and $H_1 \wedge H_2 = 0 \Rightarrow$ $\Rightarrow \bar{A}_1 \wedge A_2 = \bar{A}_2 \wedge A_1 = 0$. Hence $A_1\bar{\delta}A_2 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$ and $B_1\bar{\delta}B_2 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$ and $\bar{B}_1 \wedge B_2 = B_1 \wedge \bar{B}_2 = 0 \Rightarrow$ there exist B_1, B_2 with $A_i\bar{\delta}cB_i$, $\bar{A}_1 < \bar{B}_1 < cB_2\bar{\delta}A_2$, $\bar{A}_2 < \bar{B}_2 < cB_1\bar{\delta}A_1 \Rightarrow \bar{A}_1\bar{\delta}A_2$ and $\bar{A}_2\bar{\delta}A_1$.

Proof of (5). Let $A_0 > 0$ and $F_1 > 0$ be any two elements such that (i) F_1 is closed and (ii) $A_0 \wedge F_1 = 0$. Set $cF_1 = H_1$. Then $A_0 < H_1 = \Sigma(U : U\bar{\delta}F_1)$. Therefore there exists an open element H_0 such that

$$A_0 \wedge H_0 > 0 \quad \text{and} \quad H_0\bar{\delta}F_1.$$

Now $H_0\bar{\delta}F_1 \Rightarrow \bar{H}_0\bar{\delta}F_1 \Rightarrow$ there exist open elements G_1, G_2 with $\bar{H}_0 < G_1$, $F_1 < G_2$, $\bar{H}_0\bar{\delta}cG_1$, and $G_1\bar{\delta}G_2 \Rightarrow$ there exists an open element G_1 such that $\bar{H}_0 < G_1 < \bar{G}_1 < H_1$, $\bar{G}_1\bar{\delta}F_1$, and $\bar{H}_0\bar{\delta}cG_1$.

Setting $G_1 = H_{\frac{1}{2}}$ we have

(1) $$\bar{H}_0 < H_{\frac{1}{2}} < \bar{H}_{\frac{1}{2}} < H_1,$$

(2) $$\bar{H}_0\bar{\delta}cH_{\frac{1}{2}}, \quad \text{and} \quad (3) \quad \bar{H}_{\frac{1}{2}}\bar{\delta}F_1.$$

Replacing A_0 and F_1 by H_0 and $cH_{\frac{1}{2}}$ we can construct an open element $H_{\frac{1}{4}}$ such that (1) $\bar{H}_0 < H_{\frac{1}{4}} < \bar{H}_{\frac{1}{4}} < H_{\frac{1}{2}}$ and (2) $\bar{H}_0\bar{\delta}cH_{\frac{1}{4}}$.

We can construct H_t for every dyadic fraction $t=\frac{m}{2^n}$ by induction on n as follows.

Having defined H_t for $t=\frac{m}{2^n}$ $(m=0, 1, 2, \ldots, 2^n)$, we define $H_{t'}$ for $t'=\frac{m}{2^{n+1}}$. If m is even, $\frac{m}{2^{n+1}}=\frac{m'}{2^n}$ and $H_{\frac{m'}{2^n}}$ is defined. Let m be odd. Then $H_{\frac{m-1}{2^{n+1}}}$ and $H_{\frac{m+1}{2^{n+1}}}$ are already defined satisfying

(i) $\overline{H_{\frac{m-1}{2^{n+1}}}} < H_{\frac{m+1}{2^{n+1}}}$ and (ii) $\overline{H_{\frac{m-1}{2^{n+1}}}}\,\bar{\delta}c\left(H_{\frac{m+1}{2^{n+1}}}\right)$.

Now as before we can construct an open element G such that

(i) $$\overline{H_{\frac{m-1}{2^{n+1}}}} < G < \bar{G} < H_{\frac{m+1}{2^{n+1}}},$$

(ii) $$\overline{H_{\frac{m-1}{2^{n+1}}}}\,\bar{\delta}cG,$$

(iii) $$\bar{G}\,\bar{\delta}\,c\left(H_{\frac{m+1}{2^{n+1}}}\right).$$

Set $G=H_{\frac{m}{2^{n+1}}}$. Thus for each dyadic rational t we can define an open element H_t in $\mathfrak{B}$ such that

(i) $\bar{H}_0<H_t<\bar{H}_t<H_{t'}<\bar{H}_{t'}<H_1$ for every $t<t'$;

(ii) $A_0\wedge H_0>0$ and (iii) $F_1\wedge H_1=0$.

Hence $\mathfrak{B}$ is completely regular.

Proposition 2. *Every classical topological completely regular Boolean algebra $(\mathfrak{B}, \tau)$ is a topological proximity Boolean algebra.*

Proof. $(\mathfrak{B}, \tau)$ is uniformisable (cf. Nöbeling p. 195). Let Ω be the index set corresponding to a uniformity $\mathfrak{A}$ of $\mathfrak{B}$. Then for each element $A\in\mathfrak{B}$ and to each index $\alpha\in\Omega$ an element $V_\alpha(A)$ in $\mathfrak{B}$ is uniquely associated satisfying the uniformity axioms U_1 to U_6 (cf. Nöbeling, p. 169).

Define for any two elements A, B in $\mathfrak{B}$, $A\bar{\delta}B \Leftrightarrow$ there exists an $\alpha\in\Omega$ such that $V_\alpha(A)\wedge V_\alpha(B)=0$.

We shall show that $\bar{\delta}$ is a proximity relation on $\mathfrak{B}$ compatible with the topology τ. Axioms P_1, P_2, P_3 and P_4 follow easily from axioms U_1-U_6. Suppose $A_1\bar{\delta}A_2$. Then there exists an $\alpha\in\Omega$ such that $V_\alpha(A_1)\wedge V_\alpha(A_2)=0$. Given $\alpha\in\Omega$ there exists a $\beta\in\Omega$ such that $V_\beta(V_\beta(V_\beta(A)))\leqq V_\alpha(A)$ for each $A\in\mathfrak{B}$. Let $B_i=V_\beta(V_\beta(A_i))$ for $i=1, 2$. Then $V_\beta(B_1)\wedge V_\beta(B_2)\leqq(V_\alpha(A_1))\wedge V_\alpha(A_2)=0$. This implies $B_1\bar{\delta}B_2$. Again $B_i\wedge cB_i=0$ $\Rightarrow V_\beta(V_\beta(A_i))\wedge cB_i=0\Rightarrow V_\beta(A_i)\wedge V_\beta(cB_i)=0\Rightarrow A_i\bar{\delta}cB_i$. Thus axiom P_5 is also satisfied.

Now we shall show that an element $A\in(\mathfrak{B}, \tau)$ is open if and only if $A=\Sigma(U|U\bar{\delta}cA)$.

Suppose A is open. Then cA is closed and therefore $cA=\bigwedge_{\alpha\in\Omega} V_\alpha(cA)$. This implies $A=\sum_{d\in\Omega} c(V_d(cA))$.

$V_\alpha(cA)\wedge c(V_\alpha(cA))=0$ implies $cA\bar{\delta}c(V_\alpha(cA))$. This in turn implies

$$A=\sum_{\alpha\in\Omega} c(V_\alpha(cA))\leqq\Sigma(U|U\bar{\delta}cA).$$

Conversely suppose $A=\Sigma(U|U\bar{\delta}cA)$. Then $A=\Sigma(U|U\wedge V_\alpha(cA)=0$ for some $\alpha\in\Omega)$. This implies $A\wedge(\bigwedge_{\alpha\in\Omega} V_\alpha(cA))=0$ and therefore $A\wedge(\overline{cA})=0$. Therefore $\overline{cA}=cA$. Hence cA is closed and A is open.

Propositions 1 and 2 lead to the following result:

Proposition 3. *A topological Boolean algebra* $(\mathfrak{B},\tau)$ *is completely regular if and only if there exists a proximity relation* $\bar{\delta}$ *compatible with the topology of* $\mathfrak{B}$.

Let $(\mathfrak{B},\tau,\bar{\delta})$ be a topological proximity algebra. We call a finite covering $(A_1, A_2, \dots A_n)$ of $\mathfrak{B}$ a $\bar{\delta}$ covering or a proximity covering if there exists another covering $(B_1, B_2,\dots, B_n)$ of $\mathfrak{B}$ such that $B_i\bar{\delta}cA_i$ or $B_i\ll A_i$ for $i=1, 2, \dots, n$. Here by a covering we mean a set of elements whose Boolean sum is the unit element of $\mathfrak{B}$.

The following properties of $\bar{\delta}$-coverings are easily proved. (i) If $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are two $\bar{\delta}$-coverings of a proximity topological algebra $(\mathfrak{B},\tau,\bar{\delta})$ then the covering $\mathfrak{A}_\alpha\wedge\mathfrak{A}_\beta=(A_1\wedge A_2|A_1\in\mathfrak{A}_\alpha, A_2\in\mathfrak{A}_\beta)$ is also a $\bar{\delta}$-covering, and (ii) if $\mathfrak{A}=(A_1, A_2,\dots, A_n)$ is a $\bar{\delta}$-covering of $(\mathfrak{B},\tau,\bar{\delta})$ then $c(\sum_{i\in I} A_i)\ll\sum_{i\notin I} A_i$.

The concept of a $\bar{\delta}$-covering of a topological proximity algebra $(\mathfrak{B},\tau,\bar{\delta})$ is the extension of the notion of proximity coverings defined in ([1]). We use this concept in the proof of the following theorem:

Proposition 4. *Let* $(\mathfrak{B},\tau)$ *be a classical topological Boolean algebra and let* $\bar{\delta}$ *be a proximity relation compatible with the topology of* $\mathfrak{B}$. *Then there exists a coarsest uniform structure* $\mathfrak{A}$ *on* $\mathfrak{B}$ *compatible with* $\bar{\delta}$.

The proof of this result runs almost parallel to the proof of the corresponding theorem on proximity spaces (cf. [1]). So we shall give only the important steps in the proof.

Let $\mathfrak{A}=(\mathbf{U}_\alpha:\alpha\in\Omega)$ be the family of all finite $\bar{\delta}$-coverings of $(\mathfrak{B},\tau,\bar{\delta})$. For each element $A\in\mathfrak{B}$ and for each $\beta\in\Omega$ define $U_\beta(A)=\Sigma(A^\beta:A^\beta\in\mathbf{U}_\beta$ with $A^\beta\wedge A\neq 0)$. This defines the coarsest uniform structure on $\mathfrak{B}$ compatible with $\bar{\delta}$ i.e. for any two elements A_1, A_2 of $\mathfrak{B}$, $A_1\bar{\delta}A_2$ if and only if $U_\alpha(A_1)\wedge U_\alpha(A_2)=0$ for some $\alpha\in\Omega$. Axioms U_1, U_2, U_4 and U_5 are easily seen to hold good.

To prove U_3 suppose $\mathbf{U}_\alpha=(A_i;\ i=1, 2, \dots n)\in\mathfrak{A}$. Let I be a subset of $(1, 2,\dots n)$ and $\mathbf{U}_{\alpha_I}$ be the covering $\mathbf{U}_{\alpha_I}=(A_I, A_{I'})$ where I' is the set complement of I in $(1, 2,\dots n)$ and $A_I=\bigvee_{i\in I} A_i$. Then we have (1) U_{α_I} is a $\bar{\delta}$-covering for each I and $U_\alpha(A)=\bigwedge_I U_{\alpha_I}(A)$ and (2) for each $\mathbf{U}_{\alpha_I}$ there exists another $\bar{\delta}$-covering $\mathbf{U}_{\beta_I}$ such that

$U_{\beta_I}(U_{\beta_I}(A)) \leqq U_{\alpha_I}(A)$. To prove (1) suppose $U_\alpha(A) \ngeqq \wedge U_{\alpha_I}(A)$. Then there exists an element $B \in \mathfrak{B}$ with $B \neq 0$, $B \wedge U_\alpha(A) = 0$ and $B < \bigwedge_I U_{\alpha_I}(A)$. Let I be the set of all indices in $i = 1, 2, \ldots, n$ such that $A \wedge A_i \neq 0$. Then $U_\alpha(A) = A_I = U_{\alpha_I}(A)$ and $B \wedge U_{\alpha_I}(A) = B \wedge U_\alpha(A) = 0$ and this contradicts $B < \bigwedge_I U_{\alpha_I}(A)$. Hence $U_\alpha(A) = \bigwedge_I U_{\alpha_I}(A)$.

Using property (ii) of $\bar{\delta}$-coverings and result (3) of Proposition 1, we can construct for each subset I of $(1, 2, \ldots, n)$ elements K_{Ii} $(i = 1, 2, 3, 4)$ such that $cA_{I'} \ll K_{I1} \ll K_{I2} \ll K_{I3} \ll K_{I4} \ll A_I$. Define $B_{I_1} = K_{I2}$, $B_{I_2} = K_{I4} \wedge cK_{I1}$, and $B_{I_3} = cK_{I3}$. Then $B_{I_1} \wedge B_{I_3} = 0$, $B_{I_1} + B_{I_2} \ll A_I$ and $B_{I_2} + B_{I_3} \leqq A_{I'}$. Now $K_{I1} \ll K_{I2} \ll K_{I3} \ll K_{I4} \Rightarrow$ there exist elements L_{I_1}, L_{I_2} such that $K_{I1} \ll L_{I_1} \ll K_{I2} \ll K_{I3} \ll L_{I_2} \ll K_{I4}$. Set $M_{I_1} = L_{I_1}$, $M_{I_2} = L_{I_2} \wedge cL_{I_1}$ and $M_{I_3} = cL_{I_2}$. Then $(M_{I_1}, M_{I_2}, M_{I_3})$ is a covering of $\mathfrak{B}$ with $M_{I_i} \ll B_{I_i}$ for $i = 1, 2, 3$. This shows that $\mathbf{U}_{\beta_I} = (B_{I_i} : i = 1, 2, 3)$ is a $\bar{\delta}$-covering of $\mathfrak{B}$. Clearly for any element $A \in \mathfrak{B}$, $U_{\beta_I}(U_{\beta_I}(A)) \leqq A_I$, or $A_{I'}$ or $A_I + A_{I'}$ and in all these cases $U_{\beta_I}(U_{\beta_I}(A)) \leqq U_{\alpha_I}(A)$. This completes the proof of (2).

Let U_β be the intersection of all the coverings U_{β_I}. Then U_β is again a $\bar{\delta}$-covering and $U_\beta(A) \leqq \bigwedge_I U_{\beta_I}(A)$. Now

$$U_\beta(U_\beta(A)) \leqq \bigwedge_I (U_{\beta_I}(\wedge U_{\beta_I}(A))) \leqq \bigwedge_I U_{\beta_I}(U_{\beta_I}(A)) \leqq \bigwedge_I U_{\alpha_I}(A) \leqq U_\alpha(A).$$

Thus we have shown that axiom U_3 is satisfied.

Before proving U_6 we shall show that the uniform structure is compatible with $\bar{\delta}$. Suppose $A_1 \bar{\delta} A_2$. Then (cA_1, cA_2) is a $\bar{\delta}$-covering $= \mathbf{U}_\alpha \in \mathfrak{A}$ and $U_\alpha(A_1) = cA_2$ and therefore $U_\alpha(A_1) \wedge A_2 = 0$. Given $U_\alpha \in \mathfrak{A}$ there exists a $U_\beta \in \mathfrak{A}$ such that $U_\beta(U_\beta(A_1)) < U_\alpha(A_1)$ and for this β, $U_\alpha(A_1) \wedge A_2 = 0 \Rightarrow U_\beta(A_1) \wedge U_\beta(A_2) = 0$. Thus $A_1 \bar{\delta} A_2 \Rightarrow U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$ for some $\alpha \in \Omega$. Conversely suppose $U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$. Let U_α be the covering $(B_i : i = 1, 2, \ldots, n)$. Then there exists a subset I of $(1, 2, \ldots n)$ such that $A_1 \leqq cB_I$ and $A_2 \leqq cB_{I'}$. By property (ii) of $\bar{\delta}$-coverings $cB_I \bar{\delta} cB_{I'}$ and therefore $A_1 \bar{\delta} A_2$. Hence $A_1 \bar{\delta} A_2 \Leftrightarrow$ there exists an $\alpha \in \Omega$ such that $U_\alpha(A_1) \wedge U_\alpha(A_2) = 0$.

To prove U_6 suppose $A \in \mathfrak{B}$. Since $\bar{\delta}$ is compatible with τ,

$$c\bar{A} = \Sigma(U | U \bar{\delta} \bar{A})$$

and therefore

$$\bar{A} = \wedge (cU | U \bar{\delta} A) = \wedge (cU | U \wedge U_\alpha(A) = 0 \quad \text{for some } \alpha \in \Sigma) \geqq \wedge U_\alpha(A).$$

This proves $\bar{A} = \wedge U_\alpha(A)$.

To complete the proof of Proposition 4 we have only to show that given any uniform structure $\mathfrak{V}$ on $\mathfrak{B}$ compatible with $\bar{\delta}$ and any $\alpha \in \Omega$ there exists a $V \in \mathfrak{V}$ such that $V(A) \leqq U_\alpha(A)$ for all $A \in \mathfrak{B}$. Suppose $\mathbf{U}_\alpha = (A_i : i = 1, 2, \ldots n)$. Since U_α is a $\bar{\delta}$-covering there exists another covering $(B_i$: $i = 1, 2, \ldots n)$ such that $B_i \ll A_i$. Now $B_i \ll A_i \Rightarrow B_i \bar{\delta} cA_i \Rightarrow V_i(B_i) \wedge cA_i = 0$ for some $V_i \in \mathfrak{V}$. Given $(V_i$: $i = 1, 2, \ldots n) \in \mathfrak{V}$, there exists $V \in \mathfrak{V}$ such that $V(A) \leqq V_i(A)$ for $i = 1, 2, \ldots n$ and for all $A \in \mathfrak{B}$. We shall show that $V(A) \leqq U_\alpha(A)$ for all $A \in \mathfrak{B}$.

$$V(A) = \Sigma(V(A \wedge B_i) | A \wedge B_i \neq 0) \leqq \Sigma(V(B_i) | A \wedge B_i \neq 0) \leqq \Sigma(A_i | A \wedge B_i \neq 0) \leqq U_\alpha(A).$$

Now we proceed to study the problem of imbedding a topological proximity algebra in a compact regular space. We call a subalgebra $\mathfrak{B}$ of a compact regular space $S(X)$ an invariant subalgebra provided for each element $A \in \mathfrak{B}$ $U_\alpha(A) \in \mathfrak{B}$ for each $\mathbf{U}_\alpha \in \mathfrak{A}$ and $\wedge U_\alpha(A) \in \mathfrak{B}$, where $\mathfrak{A}$ is the unique uniform structure on $S(X)$.

Proposition 5. *Let $S(X)$ be a compact regular space. Then every invariant subalgebra of $S(X)$ is completely regular and therefore a proximity Boolean algebra.*

The proof is evident.

Proposition 6. *Every topological proximity algebra $(\mathfrak{B}, \tau, \bar{\delta})$ is $\bar{\delta}$-isomorphic to an invariant subalgebra of a compact regular space.*

Proof. Let $(\mathfrak{B}, \Omega)$ be the coarsest uniform structure on $\mathfrak{B}$ compatible with $\bar{\delta}$ constructed in the proof of Proposition 4.

Let M be the set of all ultrafilters $(\mathbf{F})$ in $\mathfrak{B}$. For each $A \in \mathfrak{B}$ let φA be the set of all ultrafilters in M to which A belongs. Then $\varphi A \in S(M)$. For each $\mathbf{F} \in M$ define $U_\alpha(\mathbf{F}) = \bigcap [\varphi(\mathbf{U}_\alpha(A)) | A \in \mathbf{F}]$ where $\bigcap \varphi U_\alpha(A)$ is the set intersection of the subsets $(\varphi(U_\alpha(A)) : A \in F)$ of M. With this uniformity, M is a complete uniform space (cf. Nöbeling, p. 202) and $(\mathfrak{B}, \tau, \bar{\delta})$ is $\bar{\delta}$ isomorphic to the subalgebra $(\varphi A : A \in \mathfrak{B})$ of $S(M)$.

To show that M is compact it is enough to prove that the uniform structure $\mathfrak{B}$ defined above, is totally bounded. Let $U_\alpha \in \mathfrak{B}$ correspond to the $\bar{\delta}$-covering $(A_i : i = 1, 2, \ldots n)$ of $\mathfrak{B}$. Let $(\mathbf{F}_i : i = 1, 2, \ldots n)$ be ultrafilters in M such that $A_i \in \mathbf{F}_i$. Let N be the finite subset $N = (\mathbf{F}_i : i = 1, 2, \ldots n)$ of M. Then clearly $U_\alpha(N) = M$. This completes the proof that $\mathfrak{B}$ is totally bounded and therefore M is compact.

Theorem 1. *A topological Boolean algebra $(\mathfrak{B}, \tau)$ is completely regular if and only if $(\mathfrak{B}, \tau)$ is homeomorphic to an invariant subalgebra of a compact regular space.*

The proof follows from Propositions 5 and 6.

2. Definition. Let X be a topological space of topological weight m and let I be an m-addive ideal of $S(X)$. Then we can define a topology in $S(X)/I$ as follows: an element $[A]$ in $S(X)/I$ is closed if and only if $A \equiv F \pmod{I}$ where F is a closed element of $S(X)$. (cf. Sikorski). We can call this the *quotient topology* on $S(X)/I$. Now we proceed to define and study quotient uniformity and quotient proximity in $S(X)/I$ where X is a completely regular space.

Proposition 2.1. *Let $(X, \bar{\delta})$ be a proximity space and let I be an ideal of $S(X)$. Then we can define a proximity structure in the quotient algebra $S(X)/I$ as follows: For any two elements $[A_1]$, $[A_2]$ in $S(X)/I$, $[A_1]\bar{\delta}[A_2] \Leftrightarrow$ there exist elements B_1, B_2 in $S(X)$ such that $A_i \equiv B_i \pmod{I}$ and $B_1 \bar{\delta} B_2$.*

Proof. *P. 1.* For any subset A of X, $A\bar{\delta}\varphi$ where φ is the null set and this implies $[A]\bar{\delta}[0]$ in $S(X)/I$.

P. 2. Clearly $[A_1]\bar{\delta}[A_2] \Leftrightarrow [A_2]\bar{\delta}[A_1]$.

P. 3. $[A_1] \wedge [A_2] > [0] \Rightarrow B_1 \cap B_2 \notin I$ for $B_i \equiv A_i$, $i = 1, 2$, $\Rightarrow B_1 \delta B_2 \Rightarrow [A_1]\delta[A_2]$.

P. 4. $[A]\bar{\delta}([B]+[C]) \Leftrightarrow [A]\bar{\delta}[B+C] \Leftrightarrow$ there exist A_1, B_1, C_1 such that $A \equiv A_1$ $B \equiv B_1$, $C \equiv C_1$ and $A_1\bar{\delta}(B_1+C_1) \Leftrightarrow [A]\bar{\delta}[B]$ and $[A]\bar{\delta}[C]$.

P. 5. Suppose $[A_1]\bar{\delta}[A_2]$. Then there exist B_1, B_2 in $[A_1]$, $[A_2]$ such that $B_1\bar{\delta}B_2$. This implies that there exist elements C_1, C_2 in $S(X)$ such that $B_i\bar{\delta}cC_i$ $(i=1, 2)$ and $C_1\bar{\delta}C_2$. Therefore $[A_1]\bar{\delta}[A_2] \Rightarrow$ there exist $[C_1]$, $[C_2]$ such that $[A_i]\bar{\delta}c[C_i]$ and $[C_1]\bar{\delta}[C_2]$.

Proposition 2.2. *Let $(X, \mathfrak{B})$ be a uniform space of topological weight m and let I be an m-additive ideal of $S(X)$. Then we can define a uniformity in the quotient $S(X)/I$ as follows: for each element $[A]$ in $S(X)/I$ $U_\alpha(A)=[U_\alpha(A^*)]$ where*

$$A^* = c(\Sigma(G|G \wedge A \in I,\ G \text{ open in } S(X))).$$

Proof. $A_1 \equiv A_2 \pmod{I} \Rightarrow A_1^* = A_2^*$ (cf. SIKORSKI).

U. 1. $[A] \leqq [A^*] \leqq [U_\alpha(A^*)] = U_\alpha[A]$.

U. 2. Given α, β there exists a γ such that $U_\alpha(A^*) \cap U_\beta(A^*) \supset U_\gamma(A^*)$ and for this γ, $U_\alpha[A] \wedge U_\beta[A] \geqq U_\gamma[A]$.

U. 3. Given α, there exists a γ such that $(U_\gamma \cdot U_\gamma)(A^*) \subset U_\alpha(A^*)$ in $S(X)$ and given γ there exists a β such that $\overline{U_\beta(A^*)} \subset U_\gamma(A^*)$. Now

$$U_\beta(U_\beta[A]) = U_\beta[U_\beta(A^*)] = [U_\beta(U_\beta(A^*))^*] \leqq [U_\beta(\overline{U_\beta(A^*)})] \leqq [U_\gamma(U_\gamma(A^*))] \leqq [U_\alpha(A^*)] = = U_\alpha[A].$$

U. 4. $$U_\alpha[A] \wedge [B] = [0] \Rightarrow U_\alpha(A^*) \cap B \in I \Rightarrow U_\alpha(A^*) \cap B^* = 0$$
$$\Rightarrow A^* \cap U_\alpha(B^*) = 0 \Rightarrow [A] \wedge U_\alpha[B] = [0].$$

U. 5. $[A_1] \leqq [A_2] \Rightarrow A_1^* \leqq A_2^* \Rightarrow U_\alpha(A_1^*) \leqq U_\alpha(A_2^*) \Rightarrow U_\alpha[A_1] \leqq U_\alpha[A_2]$.

U. 6. $\wedge U_\alpha[A] = \wedge [U_\alpha(A^*)] = \wedge \overline{[U_\alpha(A^*)]} = \overline{[\cap U_\alpha(A^*)]} = [\cap U_\alpha(A^*)] = [A^*] = \overline{[A]}$.

Theorem 2. *Let $(X, \mathfrak{B})$ be a uniform space of topological weight m and let I be an m-additive ideal of $S(X)$. Also let $\bar{\delta}$ be the proximity defined by $\mathfrak{B}$ on X. Then the quotient proximity on $S(X)/I$ defined by $\bar{\delta}$ (denoted by $\bar{\delta}_I$) is the same as the proximity defined in $S(X)/I$ by the quotient uniformity $\mathfrak{B}_I$ (denoted by $\bar{\delta}_{\mathfrak{B}}$).*

Proof. $[A_1]\bar{\bar{\delta}}_I[A_2] \Leftrightarrow B_1\bar{\delta}B_2$ for some $B_i \equiv A_i \pmod{I}$, $(i=1, 2,) \Rightarrow B_1^*\bar{\delta}B_2^*$ in $S(X) \Leftrightarrow U_\alpha(B_1^*) \cap U_\alpha(B_2^*) = \varphi$ for some $U_\alpha \in \mathfrak{B} \Rightarrow U_\alpha[A_1] \cap U_\alpha[A_2] = [0]$ in $S(X)/I$. $\Rightarrow [A_1]\bar{\delta}_{\mathfrak{B}}[A_2] \to$ (i). Conversely $[A_1]\bar{\delta}_{\mathfrak{B}}[A_2] \Rightarrow U_\alpha[A_1] \wedge U_\alpha[A_2] = [0]$ for some $\alpha \in \Omega \Rightarrow$ $\Rightarrow U_\alpha(A_1^*) \cap U_\alpha(A_2^*) \in I \Rightarrow A_1^* \cap U_\alpha(A_2^*) \in I \to$ (ii). Again $A_1^* \cap U_\alpha(A_2^*) \in I \Rightarrow A_1 \cap U_\alpha$ $(A_2^*) \in I \to$ (iii). Now given $\alpha \in \Omega$, there exists a $\beta \in \Omega$ such that $U_\beta(A_2^*) \leqq \text{int } U_\alpha(A_2^*)$. Therefore for this β, $A_1 \cap U_\alpha(A_2^*) \in I \Rightarrow A_1 \cap \text{int } U_\alpha(A_2^*) \in I \Rightarrow A_1^* \cap \text{int } U_\alpha(A_2^*) \in I \Rightarrow$ $\Rightarrow A_1^* \cap U_\beta(A_2^*) \in I \to$ (iv).

From (ii), (iii) and (iv) we have $[A_1]\bar{\delta}_{\mathfrak{B}}[A_2] \Rightarrow A_1^* \cap U_\beta(A_2^*) = \varphi$ for some $\beta \in \Omega \Rightarrow$ $\Rightarrow U_\gamma(A_1^*) \cap U_\gamma(A_2^*) = \varphi$ for some $\gamma \in \Sigma \Rightarrow A_1^*\bar{\delta}A_2^*$ in $S(X) \Rightarrow \overline{[A_1]}\bar{\delta}_I\overline{[A_2]}$ in $S(X)/I \Rightarrow$ $\Rightarrow [A_1]\bar{\delta}_I[A_2] \to$ (v). Hence from (i) and (v) we have $[A_1]\bar{\delta}_I[A_2] \Leftrightarrow [A_1]\bar{\delta}_{\mathfrak{B}}[A_2]$.

Theorem 3. *Let $(X, \bar{\delta})$ be a proximity space of topological weight m and let I be an m-additive ideal of $S(X)$. Also let $\mathfrak{U}$ denote the coarsest uniformity on $S(X)$ compatible with $\bar{\delta}$. Then the coarsest uniformity on $S(X)/I$ computible with $\bar{\delta}_I$ is the same as the quotient uniformity $\mathfrak{U}_I$.*

Proof. $\mathfrak{U}$ has a base consisting of surroundings of the form $U_\alpha = \bigcup_{i=1}^{n} (A_i \times A_i)$ where $(A_i \colon i = 1, 2, \ldots, n)$ is an open covering of $S(X)$. If $(A_i \colon i = 1, 2, \ldots, n)$ is a $\bar{\delta}$-covering of $(X, \bar{\delta})$ then $([A_i] \colon i = 1, 2, \ldots, n)$ is a $\bar{\delta}_I$-covering of $S(X)/I$. This implies $\mathfrak{A}_I$ is coarser than the coarsest uniformity compatible with $\bar{\delta}_I$ and therefore $\mathfrak{U}_I$ is the coarsest uniformity compatible with $\bar{\delta}_I$ in $S(X)/I$.

3. In this section we shall prove a proposition connecting the concept of proximity lattices of Švarc and our notion of proximity Boolean algebras.

Definition 5. A subset $\mathfrak{U}$ of a proximity Boolean algebra $(\mathfrak{B}, \bar{\delta})$ is called *Švarc open* if (i) for each $U \in \mathfrak{U}$, there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} cV$, and (ii) if U, V are in $\mathfrak{U}$ then the set $(W \colon U \bar{\delta} cW$ and $W \bar{\delta} cV) \subset \mathfrak{U}$.

Proposition 3.1. *Let $(\mathfrak{B}, \tau, \bar{\delta})$ be a topological proximity algebra. Then any element A of $\mathfrak{B}$ of the form $A = \Sigma(U \colon U \in \mathfrak{U})$ where $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$ is open. Conversely if A is an open element of $\mathfrak{B}$ then the set $(U \colon U \bar{\delta} cA)$ is a Švarc open set of $\mathfrak{B}$.*

Proof. Suppose $A = \Sigma(U \colon U \in \mathfrak{U})$ where $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$.

Now $U \in \mathfrak{U} \Rightarrow$ there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} cV \Rightarrow$ there exists a $V \leqq A$ such that $U \bar{\delta} cV \Rightarrow U \bar{\delta} cA \Rightarrow U \leqq \operatorname{int} A$. These imply $A \leqq \operatorname{int} A$ and hence A is open. Conversely suppose A is an open element of $(\mathfrak{B}, \tau, \bar{\delta})$. Let $\mathfrak{U} = (U \colon U \bar{\delta} cA)$. $U \bar{\delta} cA \Rightarrow$ there exists a V such that $U \bar{\delta} cV$ and $V \bar{\delta} cA \Rightarrow$ there exists a $V \in \mathfrak{U}$ such that $U \bar{\delta} cV$. This proves $\mathfrak{U}$ satisfies condition (i) of Definition 5. Clearly $\mathfrak{U}$ satisfies condition (ii) also. Hence $\mathfrak{U}$ is a Švarc open set of $\mathfrak{B}$.

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Semigroups having left or right zeroid elements

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An element μ of a semigroup S is called a left zeroid of S if for each $b \in S$ there exists $x \in S$ such that $xb = \mu$ [1]. A right zeroid of S is defined similarly. An element of S is a zeroid of S if it is both a left and right zeroid of S. We will assume that the reader is familiar with the notions of "simple semigroup" [2, p. 5] and "regular semigroup" [2, p. 26]. In this note we determine some properties of semigroups which are assumed to have one-sided zeroid elements. Semigroups having zeroid elements have been studied extensively by CLIFFORD and MILLER [1]. We begin with a simple but useful lemma, and agree that theorems and lemmas involving "left" and "right" cases will be stated and proved for the "left" case only, unless otherwise indicated.

Lemma 1. *If S is a semigroup with left zeroid μ and idempotent e, then $\mu e = \mu$.*

Proof. There exists $x \in S$ such that $xe = \mu$. Thus $\mu e = (xe)e = x(ee) = xe = \mu$.

Lemma 2. *If e is a left zeroid idempotent of a semigroup S, then eS is a group and Se is regular.*

Proof. Clearly e is a left identity of eS. Suppose $eb \in eS$. There exists $c \in S$ such that $c(eb) = e$. Thus $(ec)(eb) = ee = e$. Therefore e is a left zeroid of eS. Hence eS is a group.

It is obvious that e is a right identity of Se. Suppose $ze \in Se$. There exists $g \in S$ such that $g(ze) = e$. There exists $h \in S$ such that $hg = e$ and there exists $k \in S$ such that $kh = e$. Thus $eg = ke$. We have then $e = ee = (eg)(ze) = (ke)(ze)$. Therefore e is a left zeroid of Se. Suppose $x \in Se$. There exists $y \in Se$ such that $yx = e$. Thus $xyx = xe = x$, and Se is regular.

Theorem 1. *A simple semigroup with a left zeroid is regular if it contains an idempotent.*

Proof. Suppose S is a simple semigroup with left zeroid μ and idempotent e. Suppose $b \in S$. There exists $c \in S$ such that $cb = \mu$, and there exists $d \in S$ such that $dc = \mu$. Thus $\mu b = d\mu$. Therefore $(S\mu)b = (Sd)\mu \subseteqq S\mu$ and consequently $S\mu$ is a right ideal. Clearly then $S\mu$ is an ideal, and so $S = S\mu$ since S is simple. Thus every element of S is a left zeroid of S. Hence e is a left zeroid idempotent of S, and by Lemma 2, Se is regular. But by Lemma 1 we have $Se = (S\mu)e = S(\mu e) = S\mu = S$. This completes the proof.

CLIFFORD and MILLER [1] proved the next theorem and the results mentioned in the remark following. We will give a proof here because of the directness and simplicity of our proof.

Theorem 2. (CLIFFORD and MILLER) *If a semigroup S has both a left zeroid and a right zeroid, then every left or right zeroid of S is a zeroid of S.*

Proof. Suppose μ is a left zeroid of S and μ' is a right zeroid of S. Let $e=s\mu$, where $s(\mu\mu)=\mu$. Then e is a left zeroid of S such that $e\mu=\mu$. Similarly, if $f=\mu't$, where $(\mu'\mu')t=\mu'$, then f is a right zeroid of S and $\mu'f=\mu'$. Let $g\mu'=e$ and $\mu h=f$. Then $e=g\mu'=g(\mu'f)=(g\mu')f=ef=e(\mu h)=(e\mu)h=\mu h=f$. Thus if $c\in S$, then $c(y\mu)=\mu$ if $cy=f=e$. Therefore μ is a right zeroid of S. Similarly μ' is a left zeroid of S.

Remark. We note from the proof above that e is a zeroid idempotent of S and if U is the set of all zeroids of S, then $U=eS=Se$. From this it follows easily that U is a group with identity e. We further note that if $x\in S$, then $ex=e(ex)=$ $=(ex)e=e(xe)=xe$. (Thus S is a homogroup [3]). It is obvious that the mapping $x\to ex$ is a homomorphism of S onto eS.

Theorem 3. *Suppose S is a semigroup with left zeroid μ and $L=\{x\in S|x\mu=\mu\}$. Then L is a subsemigroup of S and each of the following conditions on L is sufficient for S to contain an idempotent e such that eS is a group and Se is regular:*

(1) *L has a left zeroid idempotent,*
(2) *L is degenerate,*
(3) *L has a right zeroid,*
(4) *L is regular,*
(5) *L is simple and contains an idempotent.*

In each case the mapping of S onto eS defined by $x\to ex$ is a homomorphism of S onto eS.

Proof. Clearly L is a semigroup. Suppose (1) holds. Let $f=t\mu$, where $t(\mu\mu)=\mu$. Then f is a left zeroid of S and $f\mu=\mu$. Let e be a left zeroid idempotent of L. If $b\in S$, there exists $c\in S$ such that $cb=f$. There exists $k\in L$ such that $kf=e$. Thus $(kc)b=$ $=kf=e$, and e is a left zeroid of S. By Lemma 2, eS is a group and Se is regular.

Suppose (2) holds. Then there exists a unique $e\in S$ such that $e\mu=\mu$. Thus e is an idempotent since $ee\mu=e\mu=\mu$. Since in the proof of the sufficiency of (1) we showed that L contains a left zeroid of S, then e is a left zeroid of S. Thus by Lemma 2, eS is a group and Se is regular.

Suppose (3) holds. Consider the element f in the proof of the sufficiency of (1). Since f is a left zeroid of S, if $x\in L$, there exists $y\in S$ such that $yx=f$. Thus $yx\mu=f\mu$, and so $y\mu=\mu$. Therefore $y\in L$ and f is a left zeroid of L. Now since L has a left zeroid and a right zeroid, then by Theorem 2, L has a zeroid, and by the remark following Theorem 2, L has a zeroid idempotent. Hence the conclusion follows from the sufficiency of (1).

Suppose (4) holds. Again we consider the element $f=t\mu$, where $t(\mu\mu)=\mu$. Since L is regular, there exists $z\in L$ such that $f=fzf$. It is obvious that zf is an idempotent. Clearly zf is a left zeroid of S since f is a left zeroid of S. Let $e=zf$. Again by Lemma 2, eS is a group and Se is regular.

Suppose (5) holds. Since we have shown that L contains a left zeroid, then the conclusion follows from Theorem 1 and the sufficiency of (4).

In each case the mapping $x \to ex$ of S onto eS is a homomorphism of S onto eS because e is the identity of the group eS. If $b \in S$ and $c \in S$, then $bc \to e(bc) = [(eb)e]c = (eb)(ec)$.

T. TAMURA [4] showed that if a semigroup S contains exactly one idempotent e, then e is a left zeroid of S if and only if e is a right zeroid of S. We need to prove the following variation of TAMURA's theorem in order to prove Theorem 5.

Theorem 4. *If S is a semigroup which contains among its idempotents one and only one left zeroid e, then e is a zeroid of S.*

Proof. By Lemma 2, Se is regular. Thus if $b \in Se$, there exists $x \in Se$ such that $b = bxb$. But every element of Se is a left zeroid of S. Thus the only idempotent of Se is e. Hence $bx = e$ since bx is an idempotent in Se, and so each element of Se has a right inverse in Se with respect to e. Clearly e is a right identity of Se. Therefore Se is a group and e is its identity. Hence if $c \in S$, there exists $de \in Se$ such that $(ce)(de) = e$. But $(ce)(de) = c[e(de)] = c(de)$. Therefore e is a right zeroid of S. Thus e is a zeroid of S.

K. ISEKI [3] defined a relation "$\leqq$" on the nonempty set of idempotents of a semigroup as follows: $e \leqq f$ provided $ef = e$. He showed that a homogroup always has a unique least idempotent. We can now prove the following theorem concerning ISEKI's relation.

Theorem 5. *Suppose S is a semigroup with a unique least idempotent e. If e is a left or right zeroid of S, then e is a zeroid of S.*

Proof. We will first prove the theorem for the case that e is a left zeroid of S. Suppose S contains an idempotent f which is a left zeroid of S. By Lemma 1, $fe = f$, and so $f \leqq e$. Hence because "$\leqq$" is transitive and e is the unique least idempotent of S, $f = e$. Therefore e is the only idempotent of S which is a left zeroid of S. By Theorem 4, e is a zeroid of S.

Next suppose that e is a right zeroid of S. Suppose S contains an idempotent f which is a right zeroid of S. By the dual of Lemma 1, $ef = f$. But $ef = e$ since $e \leqq f$. Hence $e = f$, and e is the only idempotent of S which is a right zeroid of S. By the dual of Theorem 4, e is a zeroid of S. This completes the proof.

We close with a theorem which gives a simple necessary and sufficient condition for a semigroup with a left zeroid to have a left zeroid idempotent.

Theorem 6. *A semigroup S with a left zeroid μ contains a left zeroid idempotent if and only if the equation $\mu = (\mu\mu)x$ has a solution $x \in S$.*

Proof. Suppose there exists $x \in S$ such that $\mu = (\mu\mu)x$. Let $f = \mu x$. Let $e = t\mu$, where $t(\mu\mu) = \mu$. Then e is a left zeroid of S and $e\mu = \mu$. We have $ef = e\mu x = \mu x$. Hence $ef = f$. But $ef = t\mu f = t\mu$. Thus $ef = e$. Therefore $e = f$ and f is a left zeroid idempotent of S.

Now we prove the "only if" part of the theorem. Suppose e is a left zeroid idempotent of S. We wish to show that e is a left zeroid of $S\mu$. Suppose $b\mu \in S\mu$. There exists $c \in S$ such that $c(b\mu)=e$. There exists $d \in S$ such that $dc=e$ and there exists $g \in S$ such that $gd=e$. Hence $ec=ge$. We have then $e=(ec)(b\mu)=(ge)(b\mu)= =(gcb\mu)(b\mu)$. Therefore e is a left zeroid of $S\mu$. By Lemma 2, $(S\mu)e$ is regular. But $(S\mu)e=S(\mu e)=S\mu$, by Lemma 1. Therefore $S\mu$ is regular. Clearly $\mu \in S\mu$ since μ is a left zeroid of S. Hence there exists $x \in S\mu$ such that $\mu=\mu x\mu$. But μx is an idempotent and by Lemma 1, $\mu=\mu(\mu x)$. This completes the proof.

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On abelian subgroups of an infinite 2-group

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In [4] and [7] the following proposition has been proved:

If G is an infinite locally finite group, then G contains an infinite abelian subgroup.

It may be conjectured that the conclusion of the above proposition is still true if one drops the condition of local finiteness. As a contribution to the solution of this problem we prove the following

Theorem.*) *An infinite 2-group contains at least one infinite abelian subgroup.*

Remark. It is a consequence of [2; p. 274, footnote] that not every 2-group is locally finite.

We shall prove our theorem by contradiction. Thus, assume that an infinite 2-group G exists all the abelian subgroups of which are finite. Denote by R the product of all locally nilpotent normal subgroups of G. Application of [3; Corollary (HIRSCH—PLOTKIN), p. 155] yields that R is locally nilpotent. Since for 2-groups local finiteness is equivalent to local nilpotency (cp. [6], Lemma 6, p. 54), R is locally finite.

Consider an abelian subgroup A^* of $G^* = G/R$, and let A be a subgroup of G satisfying $A/R = A^*$. Since A is an extension of a locally finite group by a locally finite group, A is locally finite (cp. [1], Finiteness Principle, p. 166). Application of [7; Lemma, p. 232] yields the finiteness of A. Hence A^* is finite. Thus, every abelian subgroup of G^* is finite. If G were equal to R, then G would be finite by [7; Lemma, p. 232] contradicting our assumption. Hence there exists an element a of order 2 in G^*. Since G^* is a 2-group and $(a \circ x)^{2^{i-1}} = a^{(i)} \circ x$ for all $x \in G^*$ and all integers $i \geqq 1$, a is a left Engel element of G^* (cp. [5], p. 584). Application of [6; Satz, p. 60] yields that $\{a^{G^*}\} \neq 1$ is a locally finite normal subgroup of G^*. Hence there exists a normal subgroup B of G with $B/R = \{a^{G^*}\}$. But B is locally finite and therefore contained in R, contradicting $\{a^{G^*}\} \neq 1$. From that contradiction follows the validity of our theorem.

*) Dr. O. KEGEL has informed the author that, independently and using a different method, he has obtained the same result.

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On a theorem of L. Rédei about complete oriented graphs

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The very first and perhaps most famous theorem about path problems of oriented graphs is that of L. RÉDEI, which reads as follows: "Every complete oriented finite graph has a hamiltonian path". (See [1], [2], [3], [4].)

However, this fails in the case of infinite oriented graphs, as it is pointed out in the Russian edition of the book of BERGE ([5] p. 123).

This paper gives a sufficient condition for an infinite complete graph to have a (one-sided infinite) hamiltonian path. The notations and definitions are those of C. BERGE. In this paper the term *graph* means always an *oriented graph*. Parallel edges (with coinciding or converse orientation) and selfloops are not permitted. Following BERGE's notation, we identify a graph with the ordered pair (X, Γ) where X is the vertex set of the graph and Γ is that multi-valued mapping of X into X which maps a vertex a into those vertices b for which an edge $\overrightarrow{ab}$ exists. If $A \subseteq X$ then let ΓA consist of those vertices b for which a vertex a exists satisfying $a \in A$ and $b \in \Gamma a$.

A sequence of vertices $(a_0, a_1, \dots, a_\nu)$ such that (a_i, a_{i+1}), $(i=0, 1, \dots, \nu-1)$ is an edge, is a path of length ν.

If such a path exists, we say that a_ν is a *consequent of order* ν of a_0 and we use the notation $a_\nu \in \Gamma^\nu a_0$. (The number ν is not unambigously determined by a_0 and a_ν but it depends on the path $(a_0, a_1, \dots, a_\nu)$. To the given vertices a, b there may be found several numbers ν such that $b \in \Gamma^\nu a$.)

The set $\hat{\Gamma} a$ is defined by

$$\hat{\Gamma} a = \{a\} \cup \Gamma a \cup \Gamma^2 a \cup \dots .$$

The path $(a_0, a_1, \dots, a_\nu)$ is *elementary* if its vertices are pairwise different. An elementary path $(a_0, a_1, \dots, a_k)$ is called *hamiltonian* if it contains all vertices of X.

A path $(a_0, a_1, \dots, a_k)$ where $a_0 = a_k$ and $a_0, a_1, \dots, a_{k-1}$ are pairwise different is a circuit. The definition of hamiltonian circuits is obvious.

A graph is *strongly connected* if for every pair of vertices a, b $(a \neq b)$, b is a consequent of a. A graph is *complete* if for every pair of vertices a, b, $(a \neq b)$ we have either $a \in \Gamma b$ or $b \in \Gamma a$.

If A is a subset of X the graph (A, Γ_A) is a subgraph of (X, Γ) where $\Gamma_A x = \Gamma x \cap A$. The definition of Γ_A^ν and $\hat{\Gamma}_A$ is obvious. For brevity, we shall often use the notation (A, Γ) instead of (A, Γ_A).

A hamiltonian path of the infinite graph (X, Γ) is a sequence of pairwise distinct vertices of (X, Γ), $(x_0, x_1, x_2, \ldots)$, $x_i \in X$ for which

a) $x_i \in \Gamma x_{i-1}$

b) $\{x_0, x_1, \ldots\} = X$.

Let (X, Γ) be arbitrary (finite or infinite) graph, and let M be a subset of X. We denote by $D(M)$ the set of all points x of M for which we have $\hat{\Gamma}_M x = M$.

It is easy to see, that if $D(M) \neq \emptyset$ then $(D(M), \Gamma_{D(M)})$ is strongly connected. Indeed, let (a, b) be an ordered pair of vertices of $D(M)$. Then there exists a path in (M, Γ_M) from a to b say $p = (a, x_1, x_2, \ldots, x_k, b)$. We have $b \in \hat{\Gamma}_M x_i$ $(i = 1, 2, \ldots, k)$. $\hat{\Gamma}_M b = M$, so $\hat{\Gamma}_M x_i = M$ i.e. $x_i \in D(M)$ $(i = 1, 2, \ldots k)$, and the path p passes in $(D(M), \Gamma_{D(M)})$.

Be (X, Γ) an infinite complete graph. We denote by $\mathfrak{M}$ the family of all subsets M of X for which $|X \setminus M| < \infty$.

Theorem. *The following conditions are together sufficient for (X, Γ) to have a hamiltonian path:*

C1. *X is countable;*

C2. *For an arbitrary subset $Q \in \mathfrak{M}$ there exist a set $N \subseteq Q$, $N \in \mathfrak{M}$ and a vertex $d \in D(N)$ such that one can find a hamiltonian path in the finite graph $((X \setminus N) \cup \{d\}, \Gamma)$ which terminates in d;*

C3. *There exists a subset $P \subset X$, $0 < |P| < \infty$ with the following properties:*

a) *The set of the points x of X for which $P \cap \hat{\Gamma} x \neq \emptyset$ is finite;*

b) *For any subset $M \in \mathfrak{M}$ we have $|D(M \setminus P)| < \infty$.*

Before proving the theorem we make some remarks about the conditions.

I. *The conditions* C1 *and* C2 *are also necessary.*

This is trivial for C1.

Now, assume that there exists a hamiltonian path in (X, Γ), say $(x_0, x_1, \ldots)$. Let $Q \in \mathfrak{M}$ i.e. $|X \setminus Q| < \infty$; let $X \setminus Q$ consist of the vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Denote by k the maximum of $i_1, i_2, \ldots, i_n$; the section $(x_0, x_1, \ldots, x_k)$ of the infinite hamiltonian path $(x_0, x_1, \ldots)$ contains all the points $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Choose $N = X \setminus \{x_0, x_1, \ldots, x_k\}$; $d = x_{k+1} \in D(N)$. It is easy to see that condition C2 is fulfilled.

A counterexample will show that condition C3 is not necessary. Let us define a graph (X, Γ) as follows:

$$X = \{1, 2, 3, 4, \ldots\}$$

and for any two vertices i, j $(i, j \in X;\ i < j)$,

$$j \in \Gamma_i, \quad \text{if } j - i = 1,$$

$$i \in \Gamma_j, \quad \text{if } j - i > 1.$$

(X, Γ) has a hamiltonian path $(1, 2, \ldots)$ but, as it is easy to see, condition C3 does not hold.

II. *$Q \in \mathfrak{M}$, $Q' \supset Q$ imply $Q' \in \mathfrak{M}$.* (Obvious.)

Lemma. *Consider an infinite complete graph (X, Γ) which satisfies the conditions* C1 *and* C2. *Then $\hat{\Gamma}a \in \mathfrak{M}$ for an arbitrary vertex $a \in X$.*

Proof. Let $X \setminus \{a\} = Q$. Evidently $Q \in \mathfrak{M}$. Thus, making use of C2, there exist a subset N, $N \in \mathfrak{M}$, $d \in D(N)$ and a hamiltonian path of $((X \setminus N) \cup \{d\}, \Gamma)$ which passes through a and terminates in d. So $\hat{\Gamma}a \supseteqq \hat{\Gamma}d \supseteqq N$ and by Remark II we have $\hat{\Gamma}a \in \mathfrak{M}$.

Proof of the theorem

1) Let (X, Γ) be an infinite complete graph for which the conditions C1, C2, C3 hold. Denote by P^* the set of all points x of X for which $P \cap \hat{\Gamma}x \neq \emptyset$. Then $P \subseteqq P^*$, and, by C3 a), we have $|P^*| < \infty$.

2) Since $X \setminus P^* \in \mathfrak{M}$, we can use condition C2. There exist N, d such that $N \subseteqq X \setminus P^*$, $N \in \mathfrak{M}$, $d \in D(N)$ and one can find a hamiltonian path π in $((X \setminus N) \cup \cup \{d\}, \Gamma)$ which terminates in d.

3) $p \in P^*$, $x \in X \setminus P^*$ *implies* $x \in \Gamma p$.

Proof. (X, Γ) is complete, hence for the pair of vertices p, x we have either $p \in \Gamma x$ or $x \in \Gamma p$. But $p \in \Gamma x$ implies $x \in P^*$, which is a contradiction.

4) Take $Q = X \setminus (N \cup P^*)$. We have $|Q| < \infty$ because $N \in \mathfrak{M}$. According to what has been said above, the hamiltonian path π of $((X \setminus N) \cup \{d\}, \Gamma)$ consists of two sections, the first of which goes through P^* and the second through $Q \cup \{d\}$ i.e.:

$$\pi = (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m, d); \; p_i \in P^*, \; q_i \in Q, \; d \in D(N).$$

5) Let r be an arbitrary point of $N \setminus D(N)$. Then $r \in \Gamma n$ for any point $n \in D(N)$ because, on account of the definition of $D(N)$, $r \in N$, $n \in D(N)$, $n \in \Gamma r$ imply $\hat{\Gamma}_N r = N$, i.e. $r \in D(N)$.

Denote by R the set of all points r of N for which $\hat{\Gamma}r \cap Q \neq \emptyset$.

6) $|R| < \infty$.

Proof. $Q = \emptyset$ implies $R = \emptyset$ i.e. $|R| = 0 < \infty$ and the statement is true. So it may be supposed that $0 < |Q| = m$. We have $X \setminus P^* \in \mathfrak{M}$ and $(X \setminus P^*) \cap P = \emptyset$ thus making use of C3 b) we obtain $|D(X \setminus P^*)| < \infty$. On the other hand, $D(X \setminus P^*) \neq \emptyset$ because $q_1 \in D(X \setminus P^*)$ (q_1 is the first point of the hamiltonian path π in Q; see 4). Indeed, $q_i \in \hat{\Gamma}q_1$ for any point $q_i \in Q$ and for an arbitrary point $n \in N$, $n \in \hat{\Gamma}q_1$ because of $d \in \hat{\Gamma}q_1$, $d \in D(N)$.

Denote $D(X \setminus P^*)$ by D'. From the definition of D' it follows that $q_j \in D'$ and $i < j$ imply $q_i \in D'$, so D' contains a whole section $(q_1, q_2, \ldots, q_{k_1})$ of π. Suppose $k_1 < m$ i.e. $Q \setminus D' \neq \emptyset$. Let $D'' = D(X \setminus (P^* \cup D'))$. Then $|D''| < \infty$ because of C3 b), and $D'' \neq \emptyset$ because $q_{k_1+1} \in D''$. Assuming that $Q \setminus (D' \cup D'') \neq \emptyset$ we can continue this procedure and so we get the subsets $D', D'', \ldots, D^{(i)}, \ldots$ where

$$D^{(i+1)} = D(X \setminus (P^* \cup D' \cup D'' \cup \ldots \cup D^{(i)})).$$

Since $|Q|<\infty$, this procedure comes to an end, at the kth step, say:

$$Q\subseteqq\bigcup_{1}^{k}D^{(i)};\ |D^{(i)}|<\infty,\ D^{(i)}\cap Q\neq\emptyset\quad(i=1,2,\dots,k).$$

In order to prove our statement it is sufficient to show, that $R\subseteqq\bigcup_{1}^{k}D^{(i)}$. Let $r\in R$ be an arbitrary point of R. By the definition of the subset R, we have $\hat{\Gamma}r\cap Q\neq\emptyset$. There exists an index i such that $D^{(i)}\cap\hat{\Gamma}r\neq\emptyset$; this implies $r\in D^{(i)}$.

7) Denote $Q\cup D(N)\cup R$ by D_1. We have $0<|D_1|<\infty$. ($D_1\neq\emptyset$ because $d\in D_1$.)

$$d'\in D_1,\, p\in P^*,\, x\in X\setminus(P^*\cup D_1)\quad\textit{imply}\quad d'\in\Gamma p;\quad x\in\Gamma p;\quad x\in\Gamma d'.$$

Proof. In 3), we proved $d'\in\Gamma p, x\in\Gamma p$. Thus, we have only to show that $x\in\Gamma d'$.

We have $D_1=Q\cup R\cup D(N)$. If $d'\in Q$ or $d'\in R$ then $d'\in\Gamma x$ would imply $x\in R$ which is impossible. The case $d'\in D(N)$ was examined in 5).

8) In the foregoing we have used only the following three properties of P^*:

a) $|P^*|<\infty$

b) $P\subseteqq P^*$

c) $p\in P^*,\ x\in X\setminus P^*$ imply $x\in\Gamma p$.

According to 6) and 7) the subset $P^*\cup D_1$ also possesses the properties a), b) and c). Taking $P^{**}=P^*\cup D_1$, the above construction (from 2) to 7)) can be applied to P^{**} instead of P^* and we get the subset D_2. The iteration of the procedure leads to the sequence of subsets $D_0=P^*, D_1, D_2, \dots, D_i, \dots$, where D_i is derived from the construction applied to

$$P^{(i-1)}=P^*\cup D_1\cup D_2\cup\dots\cup D_{i-1}.$$

Obviously the subsets $D_0, D_1, \dots$ are pairwise disjoint. From the construction it follows that $a\in D_k, b\in D_l, k<l$ imply $b\in\Gamma a$ $(k=0,1,2,\dots)$.

9) All the graphs $(D_0,\Gamma), (D_1,\Gamma), (D_2,\Gamma), \dots$ are non-empty *finite* complete ones, and so, by the theorem of RÉDEI, they have hamiltonian paths

$$p_0=(x_0, x_1, \dots, x_{n_0}),$$

$$p_1=(x_{n_0+1}, \dots, x_{n_1}),$$

$$p_2=(x_{n_1+1}, \dots, x_{n_2}),$$

$$\dots\dots\dots\dots\dots\dots$$

By linking the paths $p_0, p_1, p_2, \dots$, we get a hamiltonian path $(x_0, x_1, \dots, x_{n_0}, x_{n_0+1}, \dots)$ in the graph $\left(\bigcup_{0}^{\infty}D_i, \Gamma\right)$.

10) To prove our theorem we have only to show that $\bigcup_{0}^{\infty}D_i=X$. From the construction of the subsets $D_0, D_1, \dots$ it follows that $z\in X, \hat{\Gamma}z\cap D_k\neq\emptyset$ imply

$z \in \bigcup_0^k D_i$. Now suppose $z \in X \setminus \bigcup_0^\infty D_i$. According to the last remark, this means that $\hat{\Gamma} z \cap \left(\bigcup_0^k D_i\right) = \emptyset$ $(k=0, 1, 2, \ldots)$ which contradicts our Lemma.

* * *

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On stationary sets and regressive functions

By G. FODOR in Szeged

For every α, ω_α denotes the initial number of $\aleph_\alpha$ and $\aleph_{\mathrm{cf}(\alpha)}$ denotes the least cardinal number m such that $\aleph_\alpha$ can be expressed as the sum of m cardinal numbers $<\aleph_\alpha$. If $\mathrm{cf}(\alpha)=\alpha$ then $\aleph_\alpha$ and ω_α are said to be regular; otherwise they are singular. An ordinal number α is called a limit number if there is no β such that $\alpha=\beta+1$. We say that $\aleph_\alpha$ is a limit cardinal number if α is a limit number. Let now $W(\omega_\alpha)=\{\xi:\xi<\omega_\alpha\}$. We call a subset S of $W(\omega_\alpha)$ confinal to $W(\omega_\alpha)$ if for every $\nu\in W(\omega_\alpha)$ there is a $\mu\in S$ such that $\mu>\nu$. A subset C of $W(\omega_\alpha)$ is called closed if the limit of any fundamental sequence of elements of C belongs to C whenever this limit is smaller than ω_α. Let $M\subset W(\omega_\alpha)$. If $W(\omega_\alpha)-M$ does not contain a closed subset confinal to $W(\omega_\alpha)$ then we say that M is stationary; otherwise it is called non-stationary. We call a function $f(\gamma)$ on $M\subset W(\omega_\alpha)$ into $W(\omega_\alpha)$ regressive if for every $\gamma\in M$ the inequality $f(\gamma)<\gamma$ (and $f(0)=0$ for $0\in M$) holds.

We assume that $\mathrm{cf}(\alpha)>0$ and the set of the regular initial numbers $<\omega_{\mathrm{cf}(\alpha)}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$. We shall prove the following statements.

(i) *Every stationary subset of $W(\omega_\alpha)$ may be expressed as the sum of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets.*

BLOCH [1] has proved this statement for $\alpha=1$.

(ii) *Let S be a stationary subset of $W(\omega_\alpha)$. The set S may be expressed as the sum $\bigcup_{\eta<\omega_{\mathrm{cf}(\alpha)}} S_\eta$ of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets such that for each stationary subset $M\subset S$ there is an ordinal number $\eta_0<\omega_{\mathrm{cf}(\alpha)}$ for which $M\cap S_{\eta_0}$ is a stationary set.*

(iii) *If for every limit number $\xi\in W(\omega_1)$ there exists a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\dots<f_i(\xi)<\dots$ converging to ξ then for all but finitely many positive integers i there is a set S_i of the cardinal number $\aleph_1$ such that the set $\{\xi:f_i(\xi)=\gamma\}$ is stationary in $W(\omega_1)$ for each $\gamma\in S_i$.*

This theorem is a generalization of a theorem of B. ROTMAN [4].

By the proof of these statements we shall use the following

Theorem I. *Let ω_α be an initial number which is not confinal to ω, $\{K_\gamma\}_{\gamma<\tau}$ ($\tau\leqq\omega_{\mathrm{cf}(\alpha)}$) a sequence of the type τ of of non-empty and mutually disjoint non-stationary subset of $W(\omega_\alpha)$ and x_γ the first element of K_γ. Let us suppose that the elements x_γ are arranged according to their magnitude, i.e. $x_\gamma<x_\beta$ for $\gamma<\beta$. If the set $\{x_\gamma\}_{\gamma<\tau}$ is non-stationary and in the case $\tau=\omega_{\mathrm{cf}(\alpha)}$ confinal to $W(\omega_\alpha)$ then the set $\bigcup_{\gamma<\tau} K_\gamma$ is non-stationary.* (See [3].)

Theorem II. *Let ω_α be a regular initial number $>\omega$ and ϱ a regular limit ordinal number, $\varrho<\omega_\alpha$. The set of all ordinal numbers $\lambda<\omega_\alpha$ of the second kind which are confinal to ϱ is a stationary subset of $W(\omega_\alpha)$.* (See [2].)

Theorem III. *Let ω_α be an initial number which is not confinal to ω, M a subset of $W(\omega_\alpha)$ and $g(\gamma)$ a regressive function on M. If M is a stationary subset of $W(\omega_\alpha)$ then there exists an ordinal number $\pi<\omega_\alpha$ and a stationary subset N of M such that $g(\gamma)\leqq\pi$ for every $\gamma\in N$.* (See [3].)

First we prove with the method of G. Bloch [4] the following

Lemma 1. *If ω_α is a regular initial number with $\alpha>0$ and ω_β is a given regular initial number smaller than ω_α then every stationary subset of the set of the limit numbers $<\omega_\alpha$ which are confinal to ω_β may be expressed as the sum of $\aleph_\alpha$ mutually disjoint stationary sets.*

Proof. By Theorem II the set A of the limit numbers $<\omega_\alpha$ which are confinal to ω_β is a stationary subset of $W(\omega_\alpha)$. Let us denote by S a stationary subset of the set A.

We prove that there exists a regressive function φ on the set S with the property:

(P) *if $S=R\cup Q$ is a decomposition of S into two disjoint sets R and Q such that φ is bounded on the set R (i.e. there exists an ordinal number $\gamma<\omega_\alpha$ such that $\varphi(\xi)<\gamma$ for every $\xi\in R$), then the set Q is stationary in $W(\omega_\alpha)$.*

Since $S\subset A$ there exists for every element $\xi\in S$ an increasing sequence $\{\xi_\eta\}_{\eta<\omega_\beta}$ of the type ω_β of ordinal numbers $<\xi$ such that $\lim\limits_{\eta<\omega_\beta}\xi_\eta=\xi$. We define now on the set S a sequence $\{f_\eta\}_{\eta<\omega_\beta}$ of the type ω_β of regressive functions as follows: let $\xi\in S$ and

$$f_\eta(\xi)=\begin{cases}0 & \text{if } \xi=0,\\ \xi_\eta & \text{if is a limit number.}\end{cases}$$

We show now that there exists an ordinal number $\eta<\omega_\beta$ for which the function f_η has the property (P). Suppose on the contrary that $\eta<\omega_\beta$ but f_η does not have the property (P). Then to every ordinal number $\eta<\omega_\beta$ there corresponds a decomposition $S=R_\eta\cup Q_\eta$ of S into two disjoint sets R_η and Q_η such that the function f_η is bounded on the set R_η and the set Q_η is non-stationary in $W(\omega_\alpha)$. Thus by Theorem I the set $\bigcup\limits_{\eta<\omega_\beta}Q_\eta$ is non-stationary in $W(\omega_\alpha)$; consequently the set $W(\omega_\alpha)-\bigcup\limits_{\eta<\omega_\beta}Q_\eta$ contains a closed subset confinal to $W(\omega_\alpha)$. Since

$$W(\omega_\alpha)-\bigcup_{\eta<\omega_\beta}Q_\eta=\big(W(\omega_\alpha)-S\big)\cup\big(\bigcap_{\eta<\omega_\beta}R_\eta\big)$$

and the set $W(\omega_\alpha)-S$ does not contain a closed subset confinal to $W(\omega_\alpha)$, the set $\bigcap\limits_{\eta<\omega_\beta}R_\eta$ is confinal to $W(\omega_\alpha)$. Since for every $\eta<\omega_\beta$ the function f_η is bounded on the set $\bigcap\limits_{\eta<\omega_\beta}R_\eta$ and $\omega_\beta<\omega_\alpha$, there exists an ordinal number $\gamma<\omega_\alpha$ such that each of the functions f_η is bounded by γ, which contradicts the definition of the functions $f_\eta(\eta<\omega_\beta)$. Thus there exists a regressive function φ on the set S with the property (P).

Let H be the set of all ordinal numbers γ for which the set of the solutions ξ of the equation $\varphi(\xi)=\gamma$ is confinal to $W(\omega_\alpha)$. By Theorem III the set is non-empty. For every element $\gamma \in H$ let us denote by M_γ the set of the solutions of the equation $\varphi(\xi)=\gamma$:

$$M_\gamma = \{\xi \in S : \varphi(\xi)=\gamma\}.$$

It is clear that the sets M_γ $(\gamma \in H)$ are mutually disjoint. Put

$$M = \bigcup_{\gamma \in H} M_\gamma.$$

The set H is confinal to $W(\omega_\alpha)$. For if not, the function φ would be bounded on the set M. Consequently, since φ has the property (P), the set $S-M$ would be stationary, which is impossible since by Theorem III there exists a subset S' of $S-M$ which is confinal to $W(\omega_\alpha)$ and an ordinal number $\gamma<\omega_\alpha$ such that $\varphi(\xi)=\gamma$ for every $\xi \in S'$. Thus the set H is confinal to $W(\omega_\alpha)$. Let H' be the set of all elements γ of the set H for which the sets M_γ are non-stationary in $W(\omega_\alpha)$. Put

$$M' = \bigcup_{\gamma \in H'} M_\gamma.$$

Let us denote by x_γ the first element of the set M_γ. By Theorem III the set $\{x_\gamma\}_{\gamma \in H'}$ is non-stationary in $W(\omega_\alpha)$ since for the function

$$g(x_\gamma)=\gamma$$

the relation $g(x_\gamma)\neq g(x_\tau)$ holds if $\gamma \neq \tau$. Thus by Theorem I the set M' is non-stationary in $W(\omega_\alpha)$. Since the function φ has the property (P), it follows that φ is not bounded on the set $S-M$. Consequently the set $H-H'$ is confinal to $W(\omega_\alpha)$. Let γ_0 be the first element of the set $H-H'$ and let

$$L_\gamma = \begin{cases} M_{\gamma_0} \cup (S - \bigcup_{\gamma \in H} M_\gamma) \cup (\bigcup_{\tau \in H-H'} M_\tau) & \text{if} \quad \gamma = \gamma_0, \\ M_\gamma & \text{if} \quad \gamma \in H-H' \quad \text{and} \quad \gamma \neq \gamma_0. \end{cases}$$

It is clear, that

$$\bigcup_{\gamma<\omega_\alpha} L_\gamma$$

is a decomposition of the set S into mutually disjoint stationary sets.

Now we prove with the aid of Lemma 1 the following

Theorem 2. *If ω_α is an initial number with* $\mathrm{cf}(\alpha)>0$ *and the set of regular initial numbers* $<\omega_{\mathrm{cf}(\alpha)}$ *is non-stationary in* $W(\omega_{\mathrm{cf}(\alpha)})$ *then every stationary subset of* $W(\omega_\alpha)$ *may be expressed as the sum of* $\aleph_{\mathrm{cf}(\alpha)}$ *mutually disjoint stationary sets.*

Proof. We distinguish two cases:

$$\text{a)} \quad \mathrm{cf}(\alpha)=\alpha, \qquad \text{b)} \quad \mathrm{cf}(\alpha)<\alpha.$$

Case a). Let S be an arbitrary stationary subset of $W(\omega_\alpha)$, and $\{\varrho_\nu\}_{\nu<\tau}$ $(\tau \leqq \omega_\alpha)$ the sequence of the regular initial numbers $<\omega_\alpha$ arranged according to their magnitude. Let us denote by P_ν the set of the limit numbers $<\omega_\alpha$ which are confinal to ϱ_ν. It is clear that the sets P_ν $(\nu<\tau)$ give decomposition of the set $W(\omega_\alpha)-\{\xi+1 : \xi<\omega_\alpha\}$ into mutually disjoint sets. By our assumption the set $\{\varrho_\nu\}_{\nu<\tau}$ is non-stationary in $W(\omega_\alpha)$. Thus by Theorem I there exists an ordinal number $\nu_0 < \tau$ for which the set

$$M = S \cap P_{\nu_0}$$

is stationary in $W(\omega_\alpha)$. By Lemma 1 the set M can be expressed as the sum of $\aleph_\alpha$ mutually disjoint stationary sets $M_\mu(\mu < \omega_\alpha)$. Put

$$N_\mu = \begin{cases} M_0 \cup (S - M) & \text{if} \quad \mu = 0, \\ M_\mu \quad \text{if} \quad \mu < \omega_\alpha & \text{and} \quad \mu \neq 0. \end{cases}$$

It is clear that

$$\bigcup_{\mu < \omega_\alpha} N_\mu$$

is a decomposition of S into $\aleph_\alpha$ mutually disjoint stationary sets.

Case b). Let Z be a closed subset of $W(\omega_\alpha)$ which is confinal to $W(\omega_\alpha)$ and the elements of which are greater than $\omega_{\mathrm{cf}(\alpha)}$. Further let $Z = \{z_\gamma\}_{\gamma < \omega_{\mathrm{cf}(\alpha)}}$ be a well-ordering of the elements of Z according to their magnitude such that $\lim\limits_{\gamma < \tau} z_\gamma = z_\tau$ for every limit number $\tau < \omega_{\mathrm{cf}(\alpha)}$. Since the function $f(\xi) = z_\xi$ is increasing and continuous, $f(\xi)$ maps every subset of $W(\omega_{\mathrm{cf}(\alpha)})$ closed and confinal to $W(\omega_{\mathrm{cf}(\alpha)})$ into a subset of $W(\omega_\alpha)$ closed and confinal to $W(\omega_\alpha)$. It follows from this that the function $f(\xi)$ maps every stationary (or non-stationary) subset of $W(\omega_{\mathrm{cf}(\alpha)})$ into a stationary (or non-stationary) subset of $W(\omega_\alpha)$. Let $\{\varrho_\nu\}_{\nu < \tau}$ $(\tau \leqq \omega_{\mathrm{cf}(\alpha)})$ be the set of the regular initial numbers $< \omega_{\mathrm{cf}(\alpha)}$ arranged according to their magnitude. Let us denote by Γ_ν the set of the limit numbers $< \omega_{\mathrm{cf}(\alpha)}$ which are confinal to ϱ_ν. Put

$$Q_0 = \{z_{\xi+1} : \xi < \omega_{\mathrm{cf}(\alpha)}\}, \qquad P_\nu = \{z_\xi : \xi \in \Gamma_\nu\}.$$

It is clear that

$$\bigcup_{\nu < \tau} P_\nu \qquad (\tau \leqq \omega_{\mathrm{cf}(\alpha)})$$

is a decomposition of the set $Z - Q_0$ into mutually disjoint sets P_ν $(\nu < \tau)$. By our assumption the set $\{\varrho_\nu\}_{\nu < \tau}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$. Thus the set $\{z_{\varrho_\nu}\}_{\nu < \tau}$ is non-stationary in $W(\omega_\alpha)$. It is easy to see that the first element of the set P_ν is z_{ϱ_ν}. By Theorem I there exists an ordinal number $\nu_0 < \tau$ for which the set

$$M = S \cap P_{\nu_0}$$

is stationary in $W(\omega_\alpha)$. According to Lemma 1 the set Γ_ν can be expressed as the sum of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint in $W(\omega_{\mathrm{cf}(\alpha)})$ stationary sets. Consequently the set M can be expressed as $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint sets M_μ $(\mu < \omega_{\mathrm{cf}(\alpha)})$, stationary in $W(\omega_\alpha)$. Put

$$N_\mu = \begin{cases} M_0 \cup (S - M) & \text{if} \quad \mu = 0, \\ M_\mu \quad \text{if} \quad \mu < \omega_{\mathrm{cf}(\alpha)} & \text{and} \quad \mu \neq 0. \end{cases}$$

It is clear that

$$\bigcup_{\mu < \omega_{\mathrm{cf}(\alpha)}} N_\mu$$

is a decomposition of S into $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint sets, stationary in $W(\omega_\alpha)$.

Corollary 3. *If ω_α is an initial number with* $\mathrm{cf}(\alpha) = \gamma + 1$, *then every stationary subset of $W(\omega_\alpha)$ can be expressed as $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets.*

Proof. The set R of the regular initial numbers smaller than $\omega_{\gamma+1}$ has power $\leqq \aleph_\gamma$, consequently the set R is non-stationary in $W(\omega_{\gamma+1})$.

It follows from Theorem 2 with the aid of Theorem I the following

Theorem 4. *If S is a stationary subset of $W(\omega_\alpha)$, $cf(\alpha)>0$, and the set of the regular initial numbers $<\omega_{\mathrm{cf}(\alpha)}$ is non-stationary in $W(\omega_{\mathrm{cf}(\alpha)})$, then the set S can be expressed as the sum $\bigcup_{\eta<\omega_{\mathrm{cf}(\alpha)}} S_\eta$ of $\aleph_{\mathrm{cf}(\alpha)}$ mutually disjoint stationary sets S_η such that for each stationary subset M of S there is an ordinal number $\eta_0<\omega_{\mathrm{cf}(\alpha)}$ for which $M\cap S_{\eta_0}$ is a stationary set.*

II.

Suppose we are given, for each countable limit ordinal number ξ, a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\dots$ converging to ξ. B. Rotman [4] has proved that, for all but finitely many positive integers i, each function f_i takes $\aleph_1$ different values, $\aleph_1$ times each; i.e. there is a set S_i of power $\aleph_1$ such that the set $\{\xi:f_i(\xi)=\gamma\}$ has power $\aleph_1$ for each $\gamma\in S_i$.

We prove in this paper the following more general result.

Theorem 5. *If for every limit number $\xi\in W(\omega_1)$ there exists a sequence of ordinal numbers $f_1(\xi)<f_2(\xi)<\dots<f_i(\xi)<\dots$ converging to ξ then for all but finitely many positive integers i there is a set S_i of power $\aleph_1$ such that the set $\{\xi:f_i(\xi)=\gamma\}$ is stationary in $W(\omega_1)$ for each $\gamma\in S_i$.*

First we prove the following.

Lemma 6. *Let ω_α be a regular initial number with $\alpha>0$, S a stationary subset of $W(\omega_\alpha)$, and $f(\xi)$ a regressive function on S, then the difference $S-U$, where U is the union of those sets $\{\xi:f(\xi)=\gamma\}$ $(\gamma\in f(S))$ which are stationary in $W(\omega_\alpha)$, is non stationary in $W(\omega_\alpha)$.*

Proof. Let $\{\eta_\nu\}_{\nu<\tau}$ $(\tau\leqq\omega_\alpha)$ be the set of the ordinal numbers $\eta\in f(S)$ arranged according to their magnitude, for which the sets $K_\eta=\{\xi\in S:f(\xi)=\eta\}$ are non-stationary in $W(\omega_\alpha)$. Let us denote by ξ_ν the first element of the set K_{η_ν}. By Theorem III the set $\{\xi_\nu\}_{\nu<\tau}$ is non-stationary in $W(\omega_\alpha)$ since for the function

$$g(\xi_\nu)=\eta_\nu$$

the relation $g(\xi_\nu)\neq g(\xi_\tau)$ holds if $\nu\neq\tau$. Thus by Theorem I the set

$$K=\bigcup_{\nu<\tau} K_{\eta_\nu}$$

is non-stationary in $W(\omega_\alpha)$. Since $S-K$ is equal to the union of those sets $\{\xi:f(\xi)=\gamma\}$ $(\gamma\in f(S))$ which are stationary in $W(\omega_\alpha)$, the lemma is proved.

Proof of Theorem 5. It is clear that, for every $i<\omega$, the domain of the function f_i is the set L of the limit numbers $\xi\in W(\omega_1)$. Let us denote by L_i the union of the sets of the form $\{\xi:f_i(\xi)=\gamma\}$ $(\gamma\in f_i(S))$ which are stationary in $W(\omega_1)$. By Lemma 6, there exists a non-stationary set H_i for every $i<\omega$ such that $L_i = L-H_i$. Since $\omega<\omega_1$, the set

$$H=\bigcup_{i<\omega} H_i$$

is, by Theorem I, non-stationary in $W(\omega_1)$. Thus the set $L-H$ contains a closed subset Z which is confinal to $W(\omega_1)$. Since

$$L-\bigcup_{i<\omega} H_i = \bigcap_{i<\omega}(L-H_i),$$

the relation

$$Z\subset \bigcap_{i<\omega}(L-H_i) = \bigcap_{i<\omega} L_i$$

holds. Let $\{\lambda_\nu^{(i)}\}_{\nu<\tau^{(i)}}$ $(i<\omega)$ be the set of the elements $\lambda\in f_i(L)$ for which $K_\gamma = \{\xi\in L: f_i(\xi)=\lambda\}$ is stationary in $W(\omega_1)$. It is clear that

$$L_i = \bigcup_{\nu<\tau^{(i)}} K_{\lambda_\nu^{(i)}}.$$

We show now that the assumption $\tau^{(i)}<\omega_1$ for all i leads to a contradiction. If the inequality $\tau^{(i)}<\omega_1$ holds for every $i<\omega$ then the power of the set

$$\Gamma = \bigcup_{i<\omega}\{\lambda_\nu^{(i)}\}_{\nu<\tau^{(i)}}$$

is smaller than $\aleph_1$. In this case there is an ordinal number $\gamma<\omega_1$ which is greater than each element of the set Γ. Now if

$$\mu\in Z\subset\bigcap_{i<\omega} L_i,$$

then the relation

$$f_i(\mu)\in\Gamma$$

holds for each $i<\omega$; consequently

$$\mu=\lim f_i(\mu)\leqq\gamma.$$

It follows from this that each element $\mu\in Z$ is smaller than $\gamma+1$ which is impossible since the set Z is confinal to $W(\omega_1)$.

We show now that the relation $\tau^{(i)}=\omega_1$ holds for all but finitely many integers i. In the contrary case there would exist an increasing sequence $\{i_j\}_{j<\omega}$ of natural numbers such that $\tau^{(i_j)}<\omega_1$ for each $j<\omega$. But then we could apply to this sequence the preceding arguments thus obtaining a j for which $\tau^{(i_j)}=\omega_1$. This contradiction finishes the proof of Theorem 5.

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On a process concerning inaccessible cardinals. I

By G. FODOR in Szeged

With the aid of the process of MAHLO one can get a hierarchy of the weakly inaccessible ordinal numbers. This hierarchy is based on the class of the regular ordinal numbers. MAHLO [1] defined a function $\pi_{\mu,\nu}$ such that the range of $\pi_{\mu,\nu}$ at $\nu=0$ is the class of the regular ordinal numbers, at $\nu=1$ is the class of the weakly inaccessible ordinal numbers and at $\nu=\eta$ is the class of the π_η-numbers, etc. We replace $\pi_{\mu,0}$ by a function $\varphi(\mu)$ the range of which is the class of the strongly (or weakly) inaccessible ordinal numbers and we define a process which is in the first two steps similar to the process of MAHLO.

We assume that each ordinal number is the set of all smaller ordinal numbers and that a cardinal number is an initial number ω_α. We denote by C the class of all the ordinal numbers. For every ordinal number α, $\omega_{\mathrm{cf}(\alpha)}$ denotes the least cardinal number β such that ω_α can be expressed as the sum of β cardinal numbers smaller than ω_α. Obviously $\mathrm{cf}(\alpha)\leqq\alpha$. If $\mathrm{cf}(\alpha)=\alpha$ then ω_α is said to be regular; otherwise they are singular. An ordinal number α is called a limit number if there is no β such that $\alpha=\beta+1$. We say that ω_α is a limit cardinal number if α is a limit number. A regular limit cardinal number is called weakly inaccessible. A weakly inaccessible ω_α is called strongly inaccessible if for every β, $\beta<\alpha$ implies $2^{\omega_\beta}<\omega_\alpha$. A subset M of ω_α (or a subclass M of C) is called stationary if $\omega_\alpha-M$ (or $C-M$) does not contain a closed subset (or subclass) confinal to ω_α (or C); otherwise it is called non-stationary.

If f is a function and x is an element of the domain of f then the value of f at x is denoted by $f(x)$. We consider such functions f for which $f(x)$ is an ordinal number. We denote by Rf the range of the function f and by Rf/α the set of the values of f which are smaller than α. We call a function f on $M\subset C$ into C regressive if for every $\gamma\in M$, $\gamma\neq 0$, the inequality $f(\gamma)<\gamma$ holds, and $f(0)=0$ if $0\in M$. Let $A=\omega_\alpha$ or $A=C$, and let M be a subclass of A which is confinal to A. We call a function f on M into A strictly divergent if for every $\gamma\in A$ there is an ordinal number β such that $f(\xi)>\gamma$ for $\xi>\beta$.

In this paper we shall make reference to the following theorems.

Theorem I. *If $A=C$ or $A=\omega_\alpha$ with* $\mathrm{cf}(\alpha)>0$, *and M is a stationary subclass of A, then there is no strictly divergent function defined on M.* (See [2].)

Theorem II. *Let $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$, and let* **K** *be a class of non-empty and mutually disjoint non-stationary subclasses of A. If the class of the first elements of the classes belonging to* **K** *is non-stationary then the class* $\bigcup\mathbf{K}$ *is non-stationary.* (See [3].)

Theorem III. *If $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$, and $\{K_\gamma\}_{\gamma<\tau}$ $(\tau\in A)$ is a sequence of non-stationary subclasses of A then $\bigcup_{\gamma<\tau} K_\gamma$ is non-stationary.* (See [4].)

Clearly this theorem is a consequence of Theorem II.

Theorem IV. *Let $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$. Further let $\{K_\gamma\}_{\gamma\in A}$ be a sequence of non-empty and non-stationary subclasses of A such that the first elements x_γ $(\gamma\in A)$ of the classes K_γ $(\gamma\in A)$ are different and they are arranged in order of their magnitude i.e. $x_\gamma<x_\tau$ for $\gamma<\tau$. If the class $F=\{x_\gamma\}_{\gamma\in A}$ is non-stationary and if for every pair (γ,τ) of ordinal numbers with $\gamma<\tau$ the relation $x_\gamma\notin K_\tau$ holds, then $\bigcup_{\gamma\in A} K_\gamma$ is non-stationary.*

This is a consequence of Theorem II. Indeed, let

$$K'_\gamma=K_\gamma-\{x_\tau\}_{\tau>\gamma},$$

then the classes

$$K''_\gamma=K'_\gamma-\bigcup_{\delta<\gamma}K'_\delta$$

satisfy the conditions of Theorem II.

By the definition of the process we assume (as a sufficient condition) the following hypothesis:

(H) *The class of all the strongly (or weakly) inaccessible initial numbers is stationary.*

We assume that the strongly (or weakly) inaccessible initial numbers have been arranged in a strictly increasing sequence $\theta_0,\theta_1,\dots,\theta_\xi,\dots$. If we associate with every θ_ξ its index ξ, we obtain a strictly divergent function φ on the class of the strongly (or weakly) inaccessible numbers for which the inequality $\varphi(\gamma)\leqq\gamma$ holds. Thus it follows from the hypothesis (H) and Theorem I that the class of the fixed points of the function φ is stationary.

The process yields for the even ordinal numbers a sequence of matrices (a hyper-sequence) and for the odd ordinal numbers a matrix of matrices (a hyper-matrix).

The idea of our process can be loosely described as follows.

The 0*-th step of the process* is the strictly increasing sequence S_0 of the strongly (or weakly) inaccessible initial numbers.

The first step of the process is the matrix M_1 the rows of which will be defined recursively as follows. The 0-th row of M_1 is the sequence S_0. Let $\alpha>0$ denote a given ordinal number and suppose that we have already defined the ξ-th rows of M_1 for all $\xi<\alpha$. If $\alpha=\beta+1$ then the α-th row of M_1 is the strictly increasing sequence of all fixed points of the (strictly increasing) sequence of the elements of the β-th row of M_1. If α is a limit number then the α-th row of M_1 is the strictly increasing sequence of the elements of the intersection of all the ξ-th rows of M_1 with $\xi<\alpha$.

The class of the 0*-th elements of the rows of M_1 is stationary.* To show this, we define the matrix M'_1 with the aid of M_1 as follows. The α-th row of M is the sequence of the elements of the α-th row of M_1 which do not belong to the $(\alpha+1)$-th row of M_1. If we associate with every element of the α-th row of M'_1 its index corresponding to it in the α-th row of M_1 then we obtain a strictly divergent regressive function φ on the class of the elements of the α-th row of M_1. It follows from Theorem I that this class is non-stationary. The class of the rows of M_1' gives a decomposition of C into non-empty and mutually disjoint non-stationary subclasses of C. Therefore by Theorem II the class of the 0-th elements of the rows of M_1 is stationary.

The second step of the process is the hyper-sequence S_2 the elements of which we define recursively as follows. The 0-th element of S_2 is the matrix M_1. Let $\alpha > 0$ denote a given ordinal number and suppose that we have already defined the ξ-th matrix belonging to S_2 for all $\xi < \alpha$. We define the α-th matrix belonging to S_2 in the same way as we have defined the matrix M_1 starting from S_0 but, in the case $\alpha = \beta + 1$ we start instead of S_0 from the strictly increasing sequence of the 0-th elements of the rows of the β-th matrix belonging to S_2 and, in the case of a limit number α, from the strictly increasing sequence of the elements of the intersection of the classes of the 0-th elements of the rows of the ξ-th matrices belonging to S_2 with $\xi < \alpha$.

The class consisting of the elements $(0, 0)$ *of the matrices belonging to* S_2 *is stationary.* To show this, we define the hyper-sequence $S_2'' = \{N_0'', N_1'', \ldots\}$ with the aid of S_2 as follows. First we form the matrix N_α' starting with N_α in the same manner as we have formed M_1' starting with M_1. Thus, we obtain that the class of the elements of an arbitrary row of N_α' is non-stationary. After this we form the matrix N_α'' starting with N_α' in such a manner that we omit the rows of N_α' the 0-th elements of which belong to the 0-th row of $N_{\alpha+1}$. If we associate with the 0-th element of every row of N_α'' its index corresponding to it in the strictly increasing sequence of the 0-th elements of the rows of N_α then we obtain a strictly divergent regressive function on the class O_α of the 0-th elements of the rows of N_α''. Thus, by Theorem I the class O_α is non-stationary. Since no two different rows of N_α'' contain common elements and the class of the elements of an arbitrary row of N_α'' is non-stationary, it follows from Theorem II that the class of the elemets of the matrix N_α'' is non-stationary.

On the other hand, for every $0 < \beta < \alpha$ there corresponds to every element γ of N_α'' one, and only one row $R_\beta'(\gamma)$ of N_β' the 0-th elements of which is γ (and no element of N_α'' belongs to an N_β' with $\beta > \alpha$). Therefore, it follows from Theorem III that the union $\bigcup_{\beta<\alpha} R_\beta'(\gamma) = Q(\gamma)$ of the classes $R_\beta'(\gamma)$ $(\beta < \alpha)$ is non-stationary. In this way to every element γ of N_α'' there corresponds a non-stationary class $Q(\gamma)$ the smallest element of which is γ. Since the class of the elements of an arbitrary row of N_α'' is non-stationary, it follows from Theorem IV that the union Q of these non-stationary classes $Q(\gamma)$ corresponding to the elements γ of an arbitrary row of N_α'' is non-stationary. In this way to every row of N_α'' there corresponds a non-stationary class the smallest element of which is the 0-th element of the row. Since the class O_α of the 0-th elements of the rows of N_α'' is non-stationary, it follows from Theorem IV that the union U_α of the classes Q corresponding to the rows of N_α'' is non-stationary. In this manner to every matrix N_α'' there corresponds a non-stationary class U_α the smallest element of which is the element $(0, 0)$ of N_α''. Since the union of the classes U_α is equal to C, it follows from Theorem IV that the class of the $(0, 0)$-th elements of the matrices N_α, is stationary.

The third step of the process is the hyper-matrix $M_3 = (L_{\beta,\alpha})$ the rows of which we define recursively as follows. The 0-th row of M_3 is the hyper-sequence S_2. Let $\beta > 0$ denote a given ordinal number and suppose that we have already defined the ξ-th rows of M_3 for all $\xi < \beta$. We define the β-th row of M_3 in the same way as we have defined the hyper-sequence S_2 starting from S_0, but in the case $\beta = \gamma + 1$ we start instead of S_0 from the strictly increasing sequence of the elements $(0, 0)$ of the matrices belonging to the γ-th row of M_3 and, in the case of limit numbers α, from the strictly increasing sequence of the elements of the intersection of the classes of the elements $(0, 0)$ of the matrices belonging to the ξ-th rows of M_3 with $\xi < \beta$.

The class formed by the elements $(0, 0)$ *of those matrices which are the* 0-*th elements of the rows of* M_3, *is stationary*. To show this, we define the hyper-matrices $M_3' = (L_{\beta,\alpha}')$, $M_3'' = (L_{\beta,\alpha}'')$ and $M_3''' = (L_{\beta,\eta_\alpha^{(\beta)}}''')$ with the aid of M_3 as follows:

(1) We form the matrix $L_{\beta,\alpha}'$ starting with $L_{\beta,\alpha}$ in the same manner as we have fromed M_1' starting with M_1, thus we obtain that the class of the elements of an arbitrary row of $L_{\beta,\alpha}'$ is non-stationary.

(2) We form the matrix $L_{\beta,\alpha}''$ starting with $L_{\beta,\alpha}'$ in such a manner that we omit the rows of $L_{\beta,\alpha}'$ the 0-th elements of which belong to the 0-th row of $L_{\beta,\alpha+1}$; thus we obtain that the class of the 0-th elements of the rows of $L_{\beta,\alpha}''$ is non-stationary.

(3) We form the β-th row of M_3''' in such a manner that we keep (in the order of the elements of β-th row of M_3'') only those matrices $L_{\beta,\alpha}''$ the elements $(0, 0)$ of which belong to the 0-th row of the matrix $L_{\beta+1,0}$; thus we obtain that the class of the element $(0, 0)$ of the matrices belonging to the β-th row of M_3''' is non-stationary.

Consider now the hyper-sequence $S_\beta'' = \{L_{\beta,0}'', L_{\beta,1}'', \dots\}$. For every $0 < \gamma < \alpha$, there corresponds to every element η of $L_{\beta,\alpha}''$ one, and only one $R_\gamma'(\eta)$ of $L_{\beta,\gamma}'$ the 0-th element of which is η (and no element of $L_{\beta,\alpha}''$ belongs to an $L_{\beta,\gamma}'$ with $\gamma > \alpha$). Therefore, it follows from Theorem III that the union $\bigcup_{\gamma<\alpha} R_\gamma'(\eta) = Q'(\eta)$ of the classes $R_\gamma'(\eta)$ $(\gamma < \alpha)$ is non-stationary. Similarly, there corresponds to every element η of $L_{\beta,\alpha}$ one, and only one row $R_{\tau,\gamma}''$ of $L_{\tau,\gamma}'$, where $\tau < \beta$ and $\gamma < \eta$, the 0-th element of which is η. Theorem III implies that $\bigcup_{\tau<\beta} \bigcup_{\gamma<\eta} R_{\tau,\gamma}'' = Q''(\eta)$ is non-stationary. Put $Q(\eta) = Q'(\eta) \cup Q''(\eta)$. In this way to every element η of $L_{\beta,\alpha}''$ there corresponds a non-stationary class $Q(\eta)$ the smallest element of which is η. Since the class of the elements of an arbitrary row of $L_{\beta,\alpha}''$ is non-stationary, it follows from Theorem IV that the union Q of these non-stationary classes $Q(\eta)$ corresponding to the elements η of an arbitrary row of $L_{\beta,\alpha}''$ is non-stationary class the smallest element of which is the 0-th element of the row. Since the class of the 0-th elements of the rows of $L_{\beta,\alpha}''$ is non-stationary, it follows from Theorem IV that the union $U_{\beta,\alpha}$ of the classes Q corresponding to the rows of $L_{\beta,\alpha}''$ is non-stationary.

Since every element of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ belongs to the 0-th row of $L_{\beta,0}$, for every $\gamma < \beta$ there corresponds to every element μ of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ one, and only one element of S_γ'' the element $(0, 0)$ of which is μ (and no element of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ belongs to an element of S_γ with $\gamma > \beta$). Thus, for every $\gamma < \beta$, there corresponds to every element μ of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ one, and only one element of the sequence $U_\gamma = \{U_{\gamma,0}, U_{\gamma,1}, \dots\}$ the smallest element of which is μ. Since the class of the elements of an arbitrary row of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ is non-stationary, it follows from Theorem IV that the union of the non-stationary classes corresponding to the elements of an arbitrary row of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ is non-stationary. In this way to every row of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ there corresponds a non-stationary class the smallest element of which is the 0-th element of the row. Since the class of the 0-th elements of the rows of $L_{\beta,\eta_\alpha^{(\beta)}}'''$ is non-stationary, it follows from Theorem IV that the union $P_{\beta,\eta_\alpha^{(\beta)}}$ of these non-stationary classes corresponding to the rows of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ is non-stationary. In such a manner there are associated with $L'''_{\beta,\eta_\alpha^{(\beta)}}$ two non-stationary classes, $U_{\beta,\eta_\alpha^{(\beta)}}$ and $P_{\beta,\eta_\alpha^{(\beta)}}$. By Theorem III the union $R_{\beta,\eta_\alpha^{(\beta)}}$ of these classes is non-stationary. The smallest element of $R_{\beta,\eta_\alpha^{(\beta)}}$ is the element $(0, 0)$ of $L'''_{\beta,\eta_\alpha^{(\beta)}}$. Since the class of the elements $(0, 0)$ of the matrices belonging to the sequence $L'''_{\beta,\eta_0^{(\beta)}}, L'''_{\beta,\eta_1^{(\beta)}}, \ldots$ is non-stationary, it follows from Theorem IV that the union V_β of the classes $R_{\beta,\eta_0^{(\beta)}}, R_{\beta,\eta_1^{(\beta)}}, \ldots$ is non-stationary. The smallest element of V_β is the element $(0, 0)$ of $L_{\beta,0}$. Since the union of the classes V_β is equal to C it follows from Theorem IV that the class formed by the elements $(0, 0)$ of those matrices which are the 0-th elements of the rows of M, is stationary.

If we omit the rows of M_3''' in which the elements $(0, 0)$ of the 0-th elements agree with their indices in the increasing sequence of the elements $(0, 0)$ of the 0-th elements of the rows of M_3''' then we obtain the hyper-matrix $M_3^{(\mathrm{IV})}$. It can be proved that every strongly (or weakly) inaccessible initial number μ contained in the matrices belonging to $M_3^{(\mathrm{IV})}$ has the property that the set of the strongly (or weakly) inaccessible initial numbers smaller than μ is non-stationary in μ.

The process can be carried further; starting with the hyper-matrix M_3 we can form the hyper-sequence S_4 etc.

§ 1. The definition of the process

We assume that all the strongly (or weakly) inaccessible ordinal numbers are arranged in a strictly increasing sequence $\theta_0, \theta_1, \ldots, \theta_\mu, \ldots$ and we put $\varphi(\mu) = \theta_\mu$. We denote by $(\varphi(\mu))'$ the fixed points of the function $\varphi(\mu)$ i.e. all ordinal numbers μ, arranged according to their magnitude, for which $\varphi(\mu) = \mu$.

We define by transfinite induction the functions

$$f_0(\alpha^{(0)}),\ f_1(\alpha^{(0)}, \alpha^{(1)}),\ \ldots,\ f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)}),\ \ldots, \tag{1}$$

the variables η and $\alpha^{(\eta)}$ ranging over all ordinal numbers. Now, we define (1) as follows. We put $f(\alpha^{(0)}) = \varphi(\alpha^{(0)})$. We define the function $f_1(\alpha^{(0)}, \alpha^{(1)})$ by transfinite induction. Let

$$f_1(\alpha^{(0)}, 0) = f_0(\alpha^{(0)}),$$
$$f_1(\alpha^{(0)}, \delta+1) = (f_1(\alpha^{(0)}, \delta))',$$
$$Rf_1(\alpha^{(0)}, \lambda) = \bigcap_{\varrho<\lambda} Rf_1(\alpha^{(0)}, \varrho) \text{ for any limit number } \lambda.$$

The process by which we have constructed the function $f_1(\alpha^{(0)}, \alpha^{(1)})$ is called the first operation and we denote it by $\mathbf{\Gamma}_1$. Let now $\xi > 1$ be a given ordinal number and suppose that the operations $\mathbf{\Gamma}_\mu$ and the functions $f_\mu(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\zeta)}, \ldots, \alpha^{(\mu)})$, where $1 \leqq \mu < \xi$, have been already defined. We define the operation $\mathbf{\Gamma}_\xi$ and the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)})$ as follows. There are two possibilities:

a) ξ is an ordinal number of the first kind, i.e. $\xi = \tau + 1$;

b) ξ is an ordinal number of the second kind.

In the case a) let

$$f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}, 0) = f_\tau(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}).$$

Let now $\varkappa > 0$ be a given ordinal number and suppose that the functions $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, \dots, \alpha^{(\tau)}, \nu)$, where $0 \leqq \nu < \varkappa$, have already been defined. Then we define the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, 0, \dots, 0, \dots, 0, \varkappa)$ with the aid of the operation $\mathbf{\Gamma}_1$ as follows: Let

$$f_{\tau+1}(\alpha^{(0)}, 0, \dots, 0, \dots, 0, \varrho+1) = (f_{\tau+1}(0, \dots, 0, \dots, \alpha^{(\tau)}, \varrho))' \tag{2}$$

if $\varkappa = \varrho + 1$,

$$Rf_{\tau+1}(\alpha^{(0)}, 0, \dots, 0, \dots, 0, \varkappa) = \bigcap_{\nu<\varkappa} Rf_{\tau+1}(0, \dots, 0, \dots, \alpha^{(\tau)}, \nu) \tag{3}$$

if $\varkappa$ is a limit number,

$$f_{\tau+1}(\alpha^{(0)}, \eta+1, 0, \dots, 0, \dots, 0, \varkappa) = (f_{\tau+1}(\alpha^{(0)}, \eta, 0, \dots, 0, \dots, \varkappa))'$$

and

$$Rf_{\tau+1}(\alpha^{(0)}, \lambda, 0, \dots, 0, \dots, 0, \varkappa) = \bigcap_{\delta<\lambda} Rf_{\tau+1}(\alpha^{(0)}, \delta, 0, \dots, 0, \dots, 0, \varkappa)$$

if λ is a limit number. So we obtain the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, 0, \dots, 0, \dots, 0, \varkappa)$ and applying step by step the operations $\mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \dots, \mathbf{\Gamma}_\xi, \dots$ $(\mu < \xi)$ we obtain the functions

$$\begin{gathered} f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, 0, \dots, 0, \dots, 0, \varkappa) \\ \vdots \\ f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, 0, \dots, 0, \dots, 0, \varkappa) \\ \vdots \end{gathered}$$

i.e. the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, \dots, \alpha^{(\tau)}, \varkappa)$. Thus we have defined the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, \dots, \alpha^{(\tau)}, \varkappa)$ for all ordinal numbers $\varkappa$.

In the case b) we define the function

$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, \dots, 0)$$

as follows: Let

$$\begin{aligned} f_\xi(\alpha^{(0)}, 0, \dots, 0, \dots, 0) &= f_0(\alpha^{(0)}) \\ f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \dots, 0, \dots, 0) &= f_1(\alpha^{(0)}, \alpha^{(1)}) \\ &\vdots \\ f_\xi(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, 0, \dots, 0, \dots, 0) &= f_\mu(\alpha^{(0)}\alpha^{(1)}, \dots, \alpha^{(\mu)}). \\ &\vdots \end{aligned}$$

Let now $\varkappa > 0$ be a given ordinal number and suppose that the functions $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\mu)}, \dots, \nu)$, where $0 \leqq \nu < \varkappa$, have already been defined. Then we define the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \dots, 0, \dots, \varkappa)$ with the aid of the operation $\mathbf{\Gamma}_1$ as follows: Let

$$Rf_\xi(\alpha^{(0)}, 0, \dots, 0, \dots, \varrho+1) = \bigcap_{\mu<\xi} Rf_\xi(0, \dots, 0, \dots, \alpha^{(\mu)}, 0, \dots, 0, \dots, \varrho) \tag{4}$$

if $\varkappa = \varrho + 1$,

$$Rf_\xi(\alpha^{(0)}, 0, \dots, 0, \dots, \varkappa) = \bigcap_{\nu<\varkappa} Rf_\xi(\alpha^{(0)}, 0, \dots, 0, \dots, \nu) \tag{5}$$

if $\varkappa$ is a limit number,

$$f_\xi(\alpha^{(0)}, \eta+1, 0, \dots, 0, \dots, \varkappa) = (f_\xi(\alpha^{(0)}, \eta, 0, \dots, 0, \dots, \varkappa))'$$

and

$$Rf_\xi(\alpha^{(0)}, \lambda, 0, \dots, 0, \dots, \varkappa) = \bigcap_{\delta<\lambda} Rf_\xi(\alpha^{(0)}, \delta, 0, \dots, 0, \dots, \varkappa)$$

if λ is a limit number. In this way we obtain the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, \varkappa)$ and applying step by step the operations $\mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \ldots, \mathbf{\Gamma}_\mu, \ldots$ $(\mu<\xi)$ we obtain the functions

$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, 0, \ldots, 0, \ldots, \varkappa)$$
$$\vdots$$
$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, 0, \ldots, 0, \ldots, \varkappa)$$
$$\vdots$$

i.e. the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \varkappa)$ for all ordinal numbers $\varkappa$. The process by which we have constructed the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)})$ is called the ξ-th operation and we denote it by $\mathbf{\Gamma}_\xi$.

In this manner we have defined the functions

$$f_0(\alpha^{(0)}), f_1(\alpha^{(0)}, \alpha^{(1)}), \ldots, f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)}), \ldots$$

for the variables ξ, $\alpha^{(\xi)}$ ranging over all ordinal numbers. Consider now the functions $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\eta)})$. For $\mu+1 \leqq \eta$ let us denote by $A_{\mu,\eta}=A_{\mu,\eta}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$ the set of ordinal numbers $\alpha_\xi^{(\mu)}=\alpha_\xi^{(\mu)}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$, arranged according to their magnitude (i.e. $\alpha_\xi^{(\mu)}<\alpha_{\xi+1}^{(\mu)}$), for which

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\xi^{(\mu)}, \alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)}) > \alpha_\xi^{(\mu)}$$

and for $\mu=\eta$ let us benote by $A_{\eta,\eta}(0)$ the set of the ordinal numbers $\alpha_\xi^{(\eta)}=\alpha_\xi^{(\eta)}(0)$, arranged according to their magnitude, for which

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\xi^{(\eta)}) > \alpha_\xi^{(\eta)}.$$

Further, for $\mu+1 \leqq \eta$ let us denote by $n_{\mu,\eta}=n_{\mu,\eta}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$ the smallest ordinal number γ for which

$$\gamma = f_\eta(0, \ldots, 0, \ldots, \gamma, \alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)}),$$

and for $\mu=\eta$ let us denote by $n_{\eta\eta}(0)$ the smallest ordinal number γ for which

$$\gamma = f_\eta(0, \ldots, 0, \ldots, \gamma).$$

Let $\mu \leqq \eta$ and let $\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}$ be given ordinal numbers. We denote by

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

the values of $f_\xi(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})<\alpha$ for which the conditions

$$\begin{aligned}
\alpha_\zeta^{(0)} &= \alpha_\xi^{(0)}(\alpha_\zeta^{(1)}, \alpha_\tau^{(2)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}) \in A_{0,\eta}(\alpha_\zeta^{(1)}, \alpha_\tau^{(2)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}),\\
\alpha_\zeta^{(1)} &= \alpha_\zeta^{(1)}(\alpha_\tau^{(2)}, \alpha_\varrho^{(3)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \alpha^{(\eta)}) \in A_{1,\eta}(\alpha_\tau^{(2)}, \alpha_\varrho^{(3)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}),\\
&\ \vdots\\
\alpha_\varphi^{(\nu)} &= \alpha_\varphi^{(\nu)}(\alpha_\psi^{(\nu+1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}) \in A_{\nu,\eta}(\alpha_\psi^{(\nu+1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})\\
&\ \vdots
\end{aligned}$$

hold.

§ 2. Results

We prove now the following

Theorem 1. *If $i<\omega$ and $\alpha=n_{i,i}(0)$ then the set of the ordinal numbers of the form $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})<\alpha$ is non-stationary in α.*

Proof. We distinguish three cases:

(a) $i=0$, (b) $i=1$, (c) $i\geqq 2$.

Case (a). By definition, $n_{0,0}(0)$ is the smallest ordinal number ϱ for which $\varrho=f_0(\varrho)$. Therefore, the set of the ordinal numbers of the form $f_0(\alpha^{(0)})<\alpha$ is equal to

$$(6)\qquad M=\{f_0(\alpha_\xi^{(0)}): \alpha_\xi^{(0)}\in A_{0,0}(0) \quad \text{and} \quad \alpha_\xi^{(0)}<\alpha\}.$$

We define a function g on M as follows:

$$g(f_0(\alpha_\xi^{(0)}))=\alpha_\xi^{(0)}.$$

It is easy to see that the function g is strictly divergent and regressive on the set (6). Thus, it follows from Theorem I that the set (6) is non-stationary in α.

Case (b). By definition, $n_{1,1}(0)$ is the smallest ordinal number ϱ for which $\varrho=f_1(0, \varrho)$. Therefore, the set of the ordinal numbers of the form $f_1(\alpha^{(0)}, \alpha^{(1)})<\alpha$ is equal to

$$(7)\qquad \bigcup_{\beta<\alpha}\{f_1(\alpha_\xi^{(0)}(\beta), \beta): \quad \alpha_\xi^{(0)}(\beta)\in A_{0,1}(\beta) \quad \text{and} \quad \alpha_\xi^{(0)}(\beta)<\alpha\}.$$

We must prove that this set is non-stationary in α.

First we prove that the set $M=\{f_1(0, \beta): \beta<\alpha\}$ is non-stationary. For this reason we define a function g on M by writing $g(f_1(0, \beta))=\beta$. Since $f_1(0, \tau)$ is a strictly increasing function of the variable τ and for every ordinal number $\beta<\alpha$ the inequality $\beta<f_1(0, \beta)$ holds, it follows that g is a strictly divergent and regressive function on M. Therefore Theorem I implies that the set M is non-stationary in α.

Our next purpose is to show that, for each $\beta<\alpha$, the set

$$N(\beta)=\{f_1(\alpha_\xi^{(0)}(\beta), \beta): \alpha_\xi^{(0)}(\beta)<\alpha\}$$

is non-stationary in α. For each $\beta<\alpha$ we define a function g_β on $N(\beta)$ as follows:

$$g_\beta(f_1(\alpha_\xi^{(0)}(\beta), \beta))=\alpha_\xi^{(0)}(\beta).$$

Since for a given β the inequalities

$$\alpha_\xi^{(0)}(\beta)<\alpha_{\xi+1}^{(0)}(\beta)$$

and

$$\alpha_\xi^{(0)}(\beta)<f_1(\alpha_\xi^{(0)}(\beta), \beta)<f_1(\alpha_{\xi+1}^{(0)}(\beta), \beta)$$

hold, it follows that g_β is strictly divergent and regressive on $N(\beta)$. Therefore Theorem I implies that the set $N(\beta)$ $(\beta<\alpha)$ is non-stationary in α.

We are now ready to prove that the set (7) is non-stationary in α. Observe that the set of the first elements of the sets $N(\beta)$ with $\beta<\alpha$ is $\{f_1(0, \beta): \beta<\alpha\}$. Thus, the preceding considerations imply that the set (7) is the union of non-empty and mutually disjoint non-stationary sets (namely the sets $N(\beta)$ with $\beta<\alpha$) the set of the first elements of which is non-stationary. Therefore Theorem II implies that the set (7) is non-stationary in α. Hence, in the case (b), the proof is complete.

Case (c). Let $i \geqq 2$ be a given natural number. Denote by $\gamma(\beta)$ the value $f_i(0, \ldots, 0, \ldots, \beta)$. We begin the proof by showing the validity of the following statement.

(i) *Assume that $\beta \neq 0$. Then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta)=f_i(0, \ldots, 0, \gamma(\beta), \psi^{(j)}, \ldots, \psi^{(k)}, \ldots, \psi^{(i)})$$

for every j, $1 \leqq j \leqq i$, provided that $\psi^{(i)}<\beta$ and $\psi^{(k)}<\gamma(\beta)$ $(j \leqq k<i)$.

Since $\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \beta)$, we have

$$\gamma(\beta) \in Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \beta). \tag{8}$$

It follows from (2) and (3) that

$$\gamma \in Rf_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu) \tag{9}$$

for every $\nu<\beta$. First we show that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \nu) \tag{10}$$

for every $\nu<\beta$. If not, then there are two ordinal numbers $\nu_0<\beta$ and $\tau_0<\gamma(\beta)$ such that

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \tau_0, \nu_0).$$

By (2)

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu_0+1)=(f_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu_0))'.$$

This means that

$$\gamma(\beta) \notin Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu_0+1). \tag{11}$$

In virtue of the relation (8), we conclude that $\nu_0+1<\beta$. On the other hand it follows from (11) and the construction of $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})$ that

$$\gamma(\beta) \notin Rf_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu_0+1).$$

But this contradicts the fact that the relation (9) holds for every $\nu<\beta$ and we conclude that $\gamma(\beta)$ satisfies (10) for every $\nu<\beta$.

Let now l be a natural number for which $0<l<i$. Assume that whenever $\psi^{(i)}<\beta$ and $\psi^{(m)}<\gamma(\beta)$ $(l+1 \leqq m<i)$, then

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l+1)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)}). \tag{12}$$

Since $\gamma(\beta)=f_i(0, \ldots, 0, 0, \ldots, \gamma(\beta), \psi^{(i)})$ for every ordinal number $\psi^{(i)}<\beta$, it remains to prove that this assumption implies the equality

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)})$$

for $\psi^{(i)}<\beta$ and $\psi^{(m)}<\gamma(\beta)$, where $l \leqq m<i$.

It follows from the definition of $f_i(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(i)})$ that, for given $\psi^{(l+1)}, \dots, \psi^{(m)}, \dots, \psi^{(i)}$, the equalities

(13)
$$Rf_i(\alpha^{(0)}, 0, \dots, 0, \dots, \gamma(\beta), \psi^{(l+1)}, \dots, \psi^{(i)}) = \\ = \bigcap_{\mu<\gamma(\beta)} Rf_i(0, \dots, 0, \dots, \alpha^{(l-1)}, \mu, \psi^{(l+1)}, \dots, \psi^{(i)})$$

and

(14)
$$f_i(\alpha^{(0)}, 0, \dots, 0, \dots, \mu+1, \psi^{(l+1)}, \dots, \psi^{(i)}) = \\ = ((f_i(0, \dots, 0, \dots, \alpha^{(l+1)}, \mu, \psi^{(l+1)}, \dots, \psi^{(i)}))'$$

hold. By (12) and (13) we obtain for given $\psi^{(m)}$ $(l+1 \leqq m \leqq i)$, where $\psi^{(m)} < \gamma(\beta)$ $(l+1 \leqq m < i)$ and $\psi^{(i)} < \beta$, and for every $\mu < \gamma(\beta)$ that

(15)
$$\gamma(\beta) \in Rf_i(0, \dots, 0, \dots, \alpha^{(l-1)}, \mu, \psi^{(l+1)}, \dots, \psi^{(i)}).$$

Now we show that for given $\psi^{(m)}$ $(l+1 \leqq m \leqq i)$, where $\psi^{(m)} < \gamma(\beta)$ $(l+1 \leqq m < i)$ and $\psi^{(i)} < \beta$, and for every $\mu < \gamma(\beta)$ the ordinal number $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta) = f_i(0, \dots, 0, \dots, \gamma(\beta), \mu, \psi^{(l+1)}, \dots, \psi^{(i)}).$$

In the contrary case there are two ordinal numbers, $\mu_0 < \gamma(\beta)$ and $\tau_0 < \gamma(\beta)$, such that

$$\gamma(\beta) = f_i(0, \dots, 0, \dots, \tau_0, \mu_0, \psi^{(l+1)}, \dots, \psi^{(i)}).$$

Hence, by (14), we have

$$\gamma(\beta) \notin Rf_i(\alpha^{(0)}, 0, \dots, 0, \dots, \mu_0+1, \psi^{(l+1)}, \dots, \psi^{(i)}).$$

Consequently, by the definition of $f_i(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(i)})$,

$$\gamma(\beta) \notin Rf_i(0, \dots, 0, \dots, \alpha^{(l-1)}, \mu_0+1, \psi^{(l+1)}, \dots, \psi^{(i)}).$$

Since $\gamma(\beta)$ is a limit number, and in virtue of this $\mu_0+1 < \gamma(\beta)$, the last relation contradicts the fact that (15) holds for every $\mu < \gamma(\beta)$. Thus, we may conclude that if $0 < l < i$, $\psi^{(i)} < \beta$ and $\psi^{(m)} < \gamma(\beta)$ for $l \leqq m < i$, then

$$\gamma(\beta) = f_i(0, \dots, 0, \dots, \gamma(\beta), \psi^{(l)}, \dots, \psi^{(i)}).$$

The proof of the statement (i) is complete.

The same method can be used to prove the following statement.

(ii) *Assume that $\alpha^{(k)}, \dots, \alpha^{(i)}$ $(0 < k \leqq i)$ are given ordinal numbers and $\alpha^{(k)} \neq 0$. Then $\gamma = f_i(0, \dots, 0, \alpha^{(k)}, \dots, \alpha^{(i)})$ satisfies the equality*

$$\gamma = f_i(0, \dots, 0, \gamma, \psi^{(j)}, \dots, \psi^{(m)}, \dots, \psi^{(k)}, \alpha^{(k+1)}, \dots, \alpha^{(i)})$$

for every j $(1 \leqq j \leqq k)$, provided that $\psi^{(k)} < \alpha^{(k)}$ and $\psi^{(m)} < \gamma$ for each m $(j \leqq m \leqq k)$.

Now we proceed to prove the following statement.

(iii) *Assume that $\underline{\alpha}^{(0)}, \dots, \underline{\alpha}^{(k)}, \dots, \underline{\alpha}^{(i)}$ are given ordinal numbers, $\underline{\alpha}^{(0)} \neq 0$ and $\underline{\alpha}^{(k)} \neq 0$. Then $\gamma = f_i(\underline{\alpha}^{(0)}, 0, \dots, 0, \underline{\alpha}^{(k)}, \dots, \underline{\alpha}^{(i)})$ satisfies the equality*

$$\gamma = f_i(0, \dots, 0, \gamma, \psi^{(j)}, \dots, \psi^{(m)}, \dots, \psi^{(k)}, \underline{\alpha}^{(k+1)}, \dots, \underline{\alpha}^{(i)})$$

for every j $(1 \leqq j \leqq k)$, provided that $\psi^{(k)} < \underline{\alpha}^{(k)}$ and $\psi^{(m)} < \gamma$ for each m $(j \leqq m \leqq k)$.

Let us denote by λ the ordinal numbers $\underline{\alpha}^{(k)}$. It follows from the definition $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(k)})$ that

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \varrho+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \varrho, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\lambda = \varrho+1$, and

$$Rf_i(\alpha^{(0)}, 0, \ldots, 0, \lambda, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \bigcap_{\nu<\lambda} Rf_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}).$$

These imply that for every $\nu<\lambda$

$$\gamma \in Rf_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}). \tag{16}$$

Since

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \nu+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\nu<\lambda$ and

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \nu+2, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \nu+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\nu+1<\lambda$, the relation (16) implies that, for every $\nu<\lambda$,

$$\gamma = f_i(0, \ldots, 0, \gamma, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}).$$

Thus, by (ii), we obtain that for every $0<j\leqq k$

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}),$$

provided that $\psi^{(k)}<\underline{\alpha}^{(k)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k)$.

Now we can prove the following statement.

(iv) *Let $\{k_l\}_{l\leqq s}$ $(s\leqq i)$ be the strictly increasing sequence of the natural numbers $k\leqq i$ for which $\alpha^{(k)}\neq 0$. Assume that $k_0=0$. Then $\gamma=f_i(\alpha^{(0)}, \ldots, \alpha^{(i)})$ satisfies the equality*

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}, 0, \ldots, 0, \alpha^{(k_{l+1})}, \ldots, \alpha^{(i)})$$

for every l $(1\leqq l\leqq s)$ and for every j $(1\leqq j\leqq k_l)$, provided that $\psi^{(k_l)}<\alpha^{(k_l)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k_l)$.

Indeed, if (iv) is true for a fixed l, $1\leqq l<s$, then

$$\gamma = f_i(\gamma, 0, \ldots, 0, \alpha^{(k_l)}, 0, \ldots, 0, \alpha^{(k_l+1)}, \ldots, \alpha^{(i)}).$$

If we apply (iii) for $\underline{\alpha}^{(0)}=\gamma$ then we obtain that the number satisfies the equality

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l+1)}, 0, \ldots, 0, \alpha^{(k_l+2)}, \ldots, \alpha^{(i)})$$

for every j $(1\leqq j\leqq k_{l+1})$, provided that $\psi^{(k_l+1)}<\alpha^{(k_l+1)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k_{l+1})$. This proves the statement (iv).

Now we proceed the proof of Theorem 1 by showing that the set

$$Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha \tag{17}$$

is non-stationary in α. We define a function on $M=Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha$ by writing

$$g\big(f_i(0, \ldots, 0, \ldots, \beta)\big)=\beta.$$

Since $f_i(0, \ldots, 0, \ldots, \tau)$ is a strictly increasing function of the variable τ and for every $\beta < \alpha$ the inequality

$$\beta < f_i(0, \ldots, 0, \ldots, \beta)$$

holds, we obtain that the function g is strictly divergent and regressive on M. Therefore Theorem I implies that the set (17) is non-stationary in α.

After this we prove the following statement.

(v) *For every k, $0 < k \leqq i$, the set*

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α, where $\underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)}$ are given ordinal numbers $< \alpha$.

By definition

$$A_{k,i}(\alpha^{(k+1)}, \ldots, \alpha^{(i)}) = \{\alpha_\varrho^{(k)}: \quad f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \alpha^{(k+1)}, \ldots, \alpha^{(i)}) > \underline{\alpha}_\varrho^{(k)}\},$$

where $\alpha_\varrho^{(k)} = \alpha_\varrho^{(k)}(\alpha^{(k+1)}, \ldots, \alpha^{(i)})$ is a strictly increasing function of ϱ for given $\alpha^{(k+1)}, \ldots, \alpha^{(i)}$.

Consider now the sets

$$B_k = Rf_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(l)}, \ldots, \underline{\alpha}^{(i)})/\alpha =$$
$$= \{f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) < \alpha: \quad \alpha_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\},$$

where $0 \leqq k < i$ and $\underline{\alpha}^{(k)}$ $(k+1 \leqq l \leqq i)$ are given ordinal numbers $< \alpha$. We define the functions g_k $(k = 0, 1, \ldots, i-1)$ on the sets B_k $(k = 0, 1, \ldots, i-1)$ as follows:

$$g_k\big(f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big) = \alpha_\varrho^{(k)}.$$

It follows from the definition of the $A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ $(k \leqq i-1)$ that the inequalities

$$\alpha_\varrho^{(k)}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) < \alpha_{\varrho+1}^{(k)}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$$

and

$$\alpha_\varrho^{(k)} < f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) < f_i(0, \ldots, 0, \alpha_{\varrho+1}^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$$

hold for $k = 0, 1, \ldots, i-1$. These imply that the functions g_k $(k = 0, 1, \ldots, i-1)$ are strictly divergent and regressive on the sets B_k $(k = 0, 1, \ldots, i-1)$. Therefore Theorem I implies that the sets B_k $(k = 0, 1, \ldots, i-1)$ are non-stationary in α.

The preceding considerations show that, by given $\underline{\alpha}^{(1)}, \ldots, \alpha^{(k)}, \ldots, \underline{\alpha}^{(i)} < \alpha$, the sets

$$\begin{aligned} B_0 &= Rf_i(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha \\ &\vdots \\ B_k &= Rf_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha \\ &\vdots \end{aligned}$$

are non-stationary in α.

Suppose now that set

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha, \quad \text{where } 0 < k < i,$$

is non-stationary in α. This means that for every fixed $\underline{\alpha}_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ the set

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α. On the other hand it is easy to verify that for any two different elements $\underline{\alpha}_\varrho^{(k)}$ and $\underline{\alpha}_\sigma^{(k)}$ the sets

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

and

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}_\sigma^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

have no common elements.

Since the set of the first elements of the sets

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

with $\alpha_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ is equal to B_k we obtain from Theorem III that the union of these sets is non-stationary in α.

Thus we have proved the statement (v).

Since the set $Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha$ is non-stationary in α, we obtain from (v) that the set

$$M = Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\sigma^{(i)})/\alpha$$

is non-stationary in α.

Consider now an arbitrary element $\gamma = f_i(\underline{\alpha}^{(0)}, \ldots, \underline{\alpha}^{(i)}) \neq 0$ of M. Let $\{k_l\}_{l \leqq s}$ $(s \leqq i)$ be the strictly increasing sequence of the natural numbers k, $0 \leqq k \leqq i$, for which $\underline{\alpha}^{(k)} \neq 0$. Let us denote by n_0 the smallest natural number $l \leqq s$ for which $k_l \geqq 2$. Then the statements (i)—(iv) imply that γ satisfies the equality

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}, \underline{\alpha}^{(k_l+1)}, \ldots, \underline{\alpha}^{(i)}) \tag{18}$$

for every l $(n_0 \leqq l \leqq s)$ and for every j $(1 \leqq j \leqq k_l)$, provided that $\psi^{(k_l)} < \underline{\alpha}^{(k_l)}$ and $\psi^{(m)} < \gamma$ for each m $(j \leqq m < k_l)$.

Let us denote by $S_{l,j}$, where $n_0 \leqq l \leqq s$ and $1 \leqq j \leqq k_l$, the set of the sequences

$$(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$$

such that $\psi^{(k_l)} < \underline{\alpha}^{(k_l)}$ and $\psi^{(m)} < \gamma$ for $j \leqq m \leqq k_l$. It is clear that the power of the set $S_{l,j}$ is $\leqq \gamma^i = \gamma$.

It follows from the statement (v) that, for any element $(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$ of $S_{l,j}$, the set

$$B(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}) = Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(j-1)}, \gamma, \psi^{(j)}, \ldots, \psi^{(k_l)}, \underline{\alpha}^{(k_l+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α.

Since $\gamma < \alpha$, Theorem II implies that the set

$$B(\gamma) = \bigcup_{l \leqq s} \bigcup_{j \leqq k_l} \bigcup_{\psi^{(j)} < \gamma} \cdots \bigcup_{\psi^{(m)} < \gamma} \cdots \bigcup_{\psi^{(k_l)} < \underline{\alpha}^{(k_l)}} B(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$$

is non-stationary in α. On the other hand, by (18), the smallest element of the set $B(\gamma)$ is γ.

In this manner we have associated with every element $\gamma = f_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\sigma^{(i)})$ of M a set $B(\alpha_\xi^{(0)}, \alpha_\xi^{(i)}, \ldots, \alpha_\sigma^{(i)})$ the smallest element of which is γ.

It only remains to prove that

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. Since the set M is non-stationary in α, the sets

$$\begin{aligned} B_0 &= Rf_i(\alpha_\xi^{(0)}, \underline{\alpha}_\zeta^{(1)}, \dots, \underline{\alpha}_\sigma^{(i)})/\alpha \\ &\vdots \\ B_k &= Rf_i(0, \dots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})/\alpha \\ &\vdots \end{aligned}$$

are non-stationary in α, where $\underline{\alpha}_\zeta^{(1)}, \dots, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)}$ are fixed ordinal numbers $<\alpha$. Since B_1 is non-stationary in α and the set of the smallest element of the sets $B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \underline{\alpha}_\varphi^{(2)}, \dots, \underline{\alpha}_\xi^{(i)}) = B(\alpha_\sigma^{(i)})$ is B_1, Theorem IV implies that the set

$$\bigcup_{\alpha_\xi^{(0)}<\alpha} \bigcup_{\alpha_\zeta^{(1)}<\alpha} B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \underline{\alpha}_\varphi^{(2)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α.

Suppose now that the set

$$B(\underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)}) = \bigcup_{\alpha_\xi^{(0)}<\alpha} \cdots \bigcup_{\alpha_\varrho^{(k-1)}<\alpha} B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \dots, \alpha_\varphi^{(k-1)}, \underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α. It is easy to verify that the smallest element of the set $B(\underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$ is $f_i(0, \dots, 0, \underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$. Thus the set of the first elements of the sets $B(\alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$ with $\alpha_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$ is equal to B_k. Therefore Theorem IV implies that the set

$$\bigcup_{\alpha_\varrho^{(k)}<\alpha} B(\alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α. In other words the set

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. The proof of Theorem 1 is complete.

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B. Sz.-Nagy, Introduction to Real Functions and Orthogonal Expansions, XI+447 pages, Budapest — New York, Akadémiai Kiadó — Oxford University Press, 1965.

C'est la version anglaise d'un livre publié en hongrois (I^{e} éd. en 1954, II^{ème} éd. en 1961).

Les deux premiers chapitres traitent de questions élémentaires concernant la théorie des ensembles, la topologie de l'espace euclidien, les fonctions continues et semi-continues. Notons le soin avec lequel on a exposé le théorème de Weierstrass—Stone. Citons en outre la démonstration du théorème de Tietze et l'étude des fonctions monotones et à variation bornée.

Au 3ème chapitre on étudie la différentiation des fonctions. On donne l'exemple de van der Waerden d'une fonction continue sans dérivée en chaque point; on démontre ensuite, suivant F. Riesz, le théorème de Lebesgue concernant l'ensemble des points de dérivabilité d'une fonction monotone; à cette occasion on discute la notion d'ensemble de mesure nulle (par rapport à la mesure de Lebesgue) sans introduire la notion générale de mesure, on démontre le théorème de Lebesgue concernant l'ensemble des points de densité d'un ensemble linéaire, ainsi que le théorème de Denjoy—Young—Saks concernant les nombres dérivés de fonctions arbitraires.

Le chapitre 4 est dédié aux fonctions d'intervalle et à l'intégrale de Riemann. On définit l'intégrabilité et la differentiabilité au sens de Burkill des fonctions d'intervalle, on expose le théorème de Darboux concernant l'intégrabilité de ces fonctions, enfin on montre que si $\varphi(\alpha, \beta)$ est intégrable sur (a, b) alors $\varphi(\alpha, \beta)$ et son intégrale indéfinie ont les mêmes nombres dérivés presque partout. L'intégrale de Riemann est définie comme un cas particulier de l'intégrale de Burkill. On étudie les opérations avec les fonctions intégrables au sens de Riemann, la notion de la fonction primitive et on donne un exemple d'une fonction bornée qui possède une primitive sans être intégrable au sens de Riemann. On définit ensuite la mesure de Jordan comme l'intégrale de Riemann de la fonction caractéristique correspondante.

Le chapitre finit avec des considérations sur le cas des fonctions de plusieurs variables.

L'intégrale de Lebesgue est introduite au chapitre 5, suivant la méthode de F. Riesz: on part des fonctions en escalier (pour lesquelles l'intégrale se définit de manière évidente), on considère les fonctions limites presque partout de suites monotones de telles fonctions (à intégrales uniformément bornées) et l'on définit l'intégrale par passage à la limite, enfin les fonctions qui sont différences de deux fonctions de ce type. On démontre ensuite les théorèmes de Beppo Levi, Lebesgue, Fatou et on expose en détail les propriétés des fonctions intégrables (continuité absolue, intégration par parties, changement de variables, différents exemples, etc.). On passe ensuite à l'étude des fonctions et des ensembles mesurables (par définition $f(x)$ est mesurable si elle est limite presque partout d'une suite de fonctions en escalier; on démontre l'équivalence avec la définition usuelle). On expose les théorèmes de Lusin et Egoroff ainsi que le théorème de Fubini.

Au 6ème chapitre on expose les intégrales de Stieltjes et de Lebesgue—Stieltjes, ainsi que l'intégrale de Lebesgue sur les espaces abstraits. On y traite de manière détaillée du théorème de Riesz sur la représentation des formes linéaires continues sur l'espace des fonctions continues ainsi que les produits dénombrables d'espaces mesurés.

Au 7ème chapitre on introduit l'espace L^2 des fonctions à carré sommable. On démontre les inégalités de Schwarz, Minkowski, le théorème de Riesz—Fischer, l'inégalité de Bessel, l'égalité de Parseval. On définit la notion générale d'espace de Hilbert et on démontre le théorème de Fréchet—Riesz concernant la représentation des formes linéaires continues dans cet espace. La seconde partie de ce chapitre contient différents exemples importants de systèmes orthogonaux et complets: le système trigonométrique, les polynomes orthogonaux par rapport à une fonction croissante μ (et l'on retrouve, pour différents choix de μ, les polynomes de Legendre, Tchébycheff, Jacobi, Hermite et Laguerre). Un paragraphe est dédié à la suite orthogonale de Haar; on expose ensuite les points fondamentaux de la théorie de la transformation de Fourier y compris le théorème de Plancherel. Enfin on considère les espaces L^p (inégalités de Hölder, Minkowski, théorème de Riesz—Fischer, forme des fonctionnelles linéaires, etc.; le chapitre finit avec la définition et quelques propriétés des espaces de Banach.

Le dernier chapitre est sur la convergence et la sommabilité des séries de Fourier. Un paragraphe introductif donne un aperçu historique ainsi que 3 exemples de problèmes physiques qui conduisent à des développements en séries de Fourier (corde vibrante, problème de Dirichlet pour le cercle, et propagation de la chaleur). Le second paragraphe expose les résultats classiques concernant la convergence des séries de Fourier (lemme de Riemann—Lebesgue, théorème de localisation de Riemann, théorèmes de convergence de Dini—Lipschitz et de Dirichlet—Jordan, exemple de Fejér d'une fonction continue à série de Fourier divergente en un point, théorème de Pringsheim sur la convergence des séries conjuguées, ainsi qu'un théorème de F. Lukács reliant les sauts d'une fonction à la convergence de sa série conjuguée. Après un introduction aux méthodes de sommation des séries numériques, on expose la sommation des séries de Fourier par la méthode des moyennes arithmétiques ainsi que par la méthode d'Abel—Poisson.

Un certain nombre d'exercices et de problèmes intéressants sont éparpilles tout au long du livre. On regrette qu'il n'y en ait pas de plus nombreux.

Ce bref aperçu ne donne qu'une idée incomplète de la qualité du livre. On ne peut conclure, sans s'arrêter pour un instant sur l'esprit dans lequel ce livre a été conçu, et le style avec lequel il a été écrit. Le style: clair, sans économies de raisonnements inutiles, mais aussi sans longueurs inutiles; bref un style prenant vivant, qui oblige le lecteur à réfléchir. L'esprit: plutôt que d'amasser le nombre le plus grand possible de résultats dans un espace donné, c'est d'essayer de dégager les idées, d'expliquer les motifs profonds de l'introduction des notions nouvelles, d'attirer l'attention sur les résultats vraiement importants.

Le livre du professeur BÉLA SZ.-NAGY pourra donc être utilisé aussi bien dans des cours que pour l'étude individuelle. Celui qui l'aura parcouru attentivement pourra aborder aisément d'autres ouvrages plus concentrés ou plus spécialisés.

G. Gussi (Bucarest)

K. Fladt, Elementarmathematik vom höheren Standpunkte aus. 4. Teil. **Elementargeometrie III. Die elementare nichteuklidische Geometrie,** 128 Seiten, Stuttgart, Ernst Klett Verlag, 1961.

Das Buch ist eine gründlich neubearbeitete und stofflich an vielen Stellen erweiterte Auflage des Buches: M. SIMON—K. FLADT, *Nichteuklidische Geometrie in elementarer Behandlung* (Beiheft zur Zeitschr. für math. und naturw. Unterricht, Bd. 10., Leipzig—Berlin, 1925).

Der Verfasser beginnt im I. Abschnitt mit einer ausführlichen, bis auf die Gegenwart fortgeführten geschichtlichen Darstellung. In diesem Abschnitt wird schon einiges Sachliche vorausgenommen und damit das Spätere entlastet. Im II. Abschnitt geschieht — unter Zugrundelegung des Archimedischen Axiomes — die Trennung der euklidischen und der beiden nichteuklidischen Geometrie, nebst der Behandlung der Grundeigenschaften der hyperbolischen Parallelen.

Im III. und IV. Abschnitt wird die von H. LIEBMANN ohne Benützung der Stetigkeit, allein in der Ebene begründete Zuordnung zwischen Spitzeck und rechtwinkligem Dreieck und die klassische Formel für den Bogen des Grenzkreises, als Grundlage der hyperbolischen Geometrie bzw. Trigonometrie und analytischen Geometrie behandelt, die dann den Stoff der V. und VII. Abschnitte bilden.

Die VIII.—X. Abschnitte sind der Raumgeometrie gewidmet, wobei die Quaternionen und Biquaternionen als Rechenhilfsmittel verwendet werden.

Die Grundtatsachen der sphärischen und elliptischen Geometrie werden in dem III. und VI. Abschnitte besprochen.

Das Buch bringt neben zahlreichen litterarischen Hinweisen die Liste der Abhandlungen von PAUL SZÁSZ über die hyperbolische Geometrie.

Das in allen Teilen mit didaktischem Gefühl und Sorgfalt geschriebene Büchlein — ebenso, wie seine Vorgänger — gibt eine vortreffliche Einführung in die nichteuklidische Geometrie.

J. Strommer (Budapest)

A. Dinghas, Minkowskische Summen und Integrale, Superadditive Mengenfunktionale, Isoperimetrische Ungleichungen (Mémorial des Sciences Mathématiques, fasc. 149), Paris, Gauthier-Villars, 1961.

Im Jahr 1935 hat LUSTERNIK gezeigt, daß die Gültigkeit des Brunn—Minkowskischen Hauptsatzes der Theorie der konvexen Körper sich auf beliebige meßbare Punktmengen erweitern läßt. Diese Entdeckung setzte grundlegende Untersuchungen von BUSEMAN, DINGHAS, HADWIGER, HENSTOCK, MACBEATH, OHMANN, E. SCHMIDT und vielen anderen im Gang, wodurch die ganze Theorie einen riesigen Aufschwung erfuhr. Durch moderne mengengeometrische Betrachtungen wurden ältere Ergebnisse in Einfachheit der Beweise und in Tragweite weit überholt. Gleichzeitig ergaben sich aber neue Probleme, von denen viele noch nicht befriedigend beantwortet werden konnten.

Eine ausgezeichnete Darstellung dieses Fragenkomplexes ist im Buch von H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Berlin—Göttingen—Heidelberg, 1957) enthalten. Der Verfasser wendet sich einem engeren Leserkreis zu. Vorliegende Monographie ist in einer bündigen, nur dem Fachmann zugänglichen Darstellungsweise gehalten. Dafür bietet sie eine auf dem Begriff eines superadditiven Mengenfunktionals beruhende, systematisch aufgebaute Theorie mit zahlreichen tiefliegenden Ergebnissen.

Das erste Kapitel ist dem Brunn—Minkowskischen Satz, das zweite seiner von LUSTERNIK, HENSTOCK und MACBEATH herrührenden Verallgemeinerungen gewidmet. Im dritten Kapitel wird der Brunn—Minkowski—Lusterniksche Satz mit Hilfe von Symmetrisierungsmethoden von neuem begründet und das isoperimetrische Problem für sehr allgemeine Punktmengen behandelt. Das letzte Kapitel enthält eine vom Verfasser stammende Verallgemeinerung des Brunn—Minkowskischen Satzes, die zu neuen Aspekten führt.

Das Buch ist unentbehrlich jedem, der in diesem anziehenden Problemenkreis weiterforschen will.

L. Fejes Tóth (Budapest)

S. Flügge, Handbuch der Physik (Encyclopedia of Physics, Bd. III/1), **Prinzipien der klassischen Mechanik und Feldtheorie** (in English), VIII+902 pages, Berlin—Göttingen—Heidelberg, Springer-Verlag, 1960.

The present volume contains the monographs "Classical Dynamics" by J. L. SYNGE and "The Classical Field Theories" by C. TRUESDELL and R. A. TOUPIN, with an appendix on "Tensor Fields" by J. L. ERICKSEN. The first monograph deals with the kinematics and dynamics of a particle, of systems of particles, and of rigid bodies; with the general transformation theory of mechanics, and finally with relativistic dynamics. Particularly interesting and modern is the treatement of the general transformation theory, the geometrical representations of dynamics in the space of events, the energy-momentum space, the space of states and energy, and the general theory of phase space. The second monograph deals with the field viewpoint in classical physics, the general theory of continuous media, singular surfaces and waves, the concept of stress, energy and entropy of systems, conservation of charge, energy and momentum, finally the constitutive equations of kinematics, energy, general mechanical and thermo-mechanical systems, the constitutive equations of electromagnetic and electromechanical systems. The appendix about tensor fields contains rather the general than the geometrical concepts of these fields. The latter two-third part of the volume is devoted to a peculiar survey of the idea of classical field theories, hardly to be found elsewhere, with detailed references on earlier work done, but less complete as regards the literature of recent years.

J. I. Horváth (Szeged)

Nicolae Dinculeanu, Intégration sur les espaces localement compacts, 592 pages, Bucarest, Editions de l'Académie, 1965.

Ce livre est consacré à l'étude détaillée et unitaire de l'intégration, par rapport à une mesure à valeurs opérateurs d'un espace de Banach E dans un autre F, des fonctions définies sur l'espace localement compact T à valeurs dans E. Son contenu est divisé en 7 chapitres: I. Mesures sur des espaces localement compacts. — II. Les espaces L^p. Fonctions intégrables. — III. Fonctions mesurables. L'espace L^∞. — IV. Mesures définies par des densités. — V. Sommes de mesures. Images.

des mesures. — VI. Mesures sur des groupes localement compacts. — VII. Espaces des champs de vecteurs.

Dans le premier chapitre on étudie les propriétés élémentaires des mesures de Radon vectorielles. Telle mesure est, d'après l'auteur, une application linéaire de l'espace des fonctions continues $f\colon T \to E$, à support compact, dans F, satisfaisant la condition habituelle de continuité.

Le chapitre II concerne surtout la théorie de l'intégration par rapport à une mesure numérique, mais se termine par l'intégration des fonctions vectorielles par rapport aux mesures opératorielles. L'extension de la mesure de Radon, bien qu'elle suive la voie de Bourbaki, diffère de celle-ci par le fait que l'auteur développe toute la théorie à partir de ce que Bourbaki appelle l'intégrale supérieure essentielle. Cela permet de donner une grande unité aux maints énoncés dans les chapitre IV et V. La mesurabilité (définie comme chez Bourbaki par le théorème de Lusin) est étudiée dans le chapitre III où on trouve aussi l'étude au relèvement dans L^{∞}-vectoriel.

Dans les chapitres IV, V et VI bien qu'on trouve tous les résultats habituels liés aux problèmes indiqués par leurs titres, une grande prépondérance est donnée au cas vectoriel ou opératoriel.

Par exemple le chapitre IV est consacré à l'étude des mesures opératorielles $n=g\cdot m$ où m est une mesure opératorielle et g une fonction vectorielle. De même dans le chapitre VI l'auteur étudie la convolution $n*m$ des mesures vectorielle n et opératorielle m définies sur le groupe G. Tout cela constitue un des côtés originels de cette excellente monographie qui peut être utile aussi bien au debutant qu'au spécialiste.

Le livre finit par l'étude des espaces L^p ou de ceux d'Orlicz formés par des champs de vecteurs.

Vu l'importance croissante que les mesures vectorielles semblent avoir dans plusieures branches des mathématiques d'aujourd'hui, ce livre sera de grande utilité.

C. Foiaş (Bucarest)

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